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Set theory interpretation for exponential approximation of time-ordered integral

Zaman-Sıralı integralin üstel fonksiyon yakınsaması için küme teorisi yorumu

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Abstract

This introductory study suggests a formal basis for the interpretation of a continuous path in a connected matrix Lie group to be represented by the set of von Neumann ordinals which is a set-theoretical interpretation of natural numbers. In this study, it is aimed to relate the discrete recurrent structure of von Neumann ordinals to the exponential function. Since the Exponential function is fundamentally integrated into science and engineering literature this work aims to discover ties between the Exponential function and sets where, the Exponential function utilized in machine learning, loss functions; cryptography, key exchange and encryption algorithms; robotics, kinematics, trajectory planning; numerical analysis, discrete integration. Thus, the set theoretical interpretation of the exponential function has an interdisciplinary critical role. Throughout the article, necessary conjectures are postulated to interpret the rotations that form a smooth curve in terms of sets, namely von Neumann ordinals. Introduced formalizations covering Set existence axiom, unit element for set groups, interpretation of a smooth curve in terms of multiplication of exponentials, introduced a derivative operator to observe limited differentiable properties of the exponential function.

Keywords: Discrete mathematics, lie group, von Neumann ordinals, smooth curve, exponential function, derivative operator, rotation group, set theory

Öz

Bu giriş çalışması, bağlı matris Lie grubundaki sürekli bir yolun, doğal sayıların küme teorik bir yorumu olan von Neumann ordinali olan kümeler ile temsil edilmesi için biçimsel bir temel önermektedir. Bu çalışmada, von Neumann ordinallerinin ayrık tekrarlayan yapısının üstel fonksiyon ile ilişkilendirilmesi amaçlanmıştır. Üstel fonksiyon temelde bilim ve mühendislik literatürüne entegre olduğundan bu çalışma, makine öğrenimi, kayıp fonksiyonları; kriptografi, anahtar değişimi ve şifreleme algoritmaları; robotik, kinematik, yörünge planlama; sayısal analiz, ayrık entegrasyon gibi alanlarda kullanılan Üstel fonksiyon ile kümeler arasındaki bağları keşfetmeyi amaçlamaktadır. Bu nedenle, üstel fonksiyonun küme teorik yorumu disiplinler arası kritik bir role sahiptir. Makale boyunca, düzgün bir eğri oluşturan rotasyonları kümeler, yani von Neumann ordinalleri açısından yorumlamak için gerekli varsayımlar öne sürülmektedir. Küme varlığı aksiyomu, küme grupları için birim eleman, düzgün bir eğrinin üstellerin çarpımı açısından yorumlanması, üstel fonksiyonun kısmen türevlenebilir özelliklerini gözlemlemek için bir türev operatörünün tanımlanmasını kapsayan formalizasyonlar tanıtılmıştır.

Anahtar kelimeler: Ayrık matematik, Lie grubu, von Neumann ordinalleri, pürüzsüz eğri, türev operatörü, rotasyon grubu, küme teorisi

1 Introduction

The von Neumann ordinals are a set-theoretical representation of natural numbers. They represent the effort to represent natural numbers in terms of sets. Where an entire set of natural numbers is generated from $\emptyset$ with a recurrence relation $V_{i+1} = V_i \cup V_i^+$ where V_i^+ simply wraps return as a set such as $\{V_i\}$. This simple recurrence relation allows the generation of a finite set of natural numbers. Considering $V_1 = \emptyset$, the first two levels are given below.

$$V_1 = V_0 \cup V_0^+ = \emptyset \cup \{\emptyset\} = \{\emptyset\} = 1 \quad (1)$$

$$V_2 = V_1 \cup V_1^+ = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} = 2 \quad (2)$$

The recurrence relation forms a resemblance with cellular automata where it generates various patterns according to simple rules [1]. In this study, we aim to relate the discrete recurrent structure of von Neumann ordinals to the exponential function. Exponential function is a pillar of Group theory and is fundamentally integrated into scientific and engineering literature. Therefore, discovering ties of exponential function with a way of interpreting natural numbers may reveal possibilities for combinatorics, numerical analysis, and group theoretic calculations used in a wide range of fields such as machine learning, statistics, cryptography, and robotics.

The exponential map holds a critical role in the group theory and in the time-ordering of derivative operators. However, in group theory, this representation is usually shown as the topology of SO(3), S2 where smooth curves are represented on the surface of the 2D sphere. A smooth curve formed of the directional derivatives on the points of the 2D sphere. For this reason, each directional derivative along the smooth curve is placed on the tangent space of a point (see Fig. 1) and projects to an infinitesimally close point on the sphere.

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The main contribution of this study is to provide a formal basis for the interpretation of a continuous path in a connected matrix Lie group to be represented by the set of von Neumann ordinals. For this purpose, this work postulates conjectures to interpret the rotations that form a smooth curve in terms of sets, namely von Neumann ordinals.

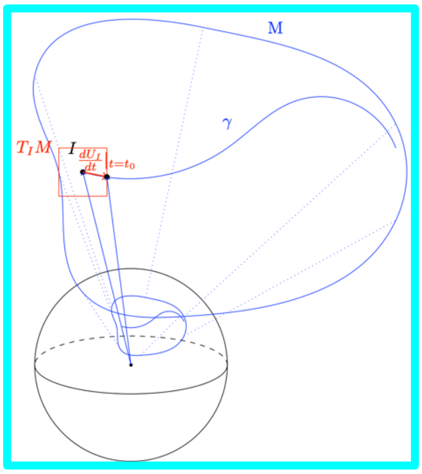

Figure 1. Smooth curve γ of Lie group on Manifold M on the SO(3). Shown with tangent space at point t_0 with a tangent vector on that point. Figure derived from the original [2].

Structure of this article; Section 2.1 describes the theoretical basis for deriving the Taylor series approximation of the Exponential function from the Binomial theorem. Section 2.2 defines conjectures and proofs.

Section 2.2 is structured as Conjecture 2.2.1, Set existence axiom is defined for unitary rotation groups; in Corollary 2.2.1 identity element defined w.r.t Conjecture 2.2.1; in Lemma 2.2.1 derivation of von Neumann ordinals and derivation of Exponential function from binomial theorem used for deriving a set expansion equivalent to exponential function based on Theorem 2.1 and Corollary 2.2.1; Conjecture 2.2.2 introduce derivative operator for the set equivalent of the Exponential function introduced in Lemma 2.2.1. Finally, concluded with proof of Conjecture 2.2.1.

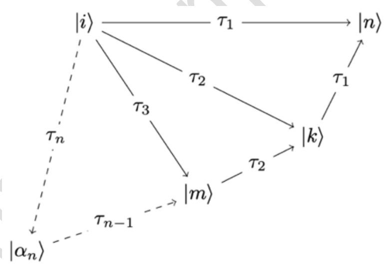

Figure 2. For a state change from |i⟩ to |n⟩, intermediate states arise from higher order approximations of U_I. Figure inspired from [4].

Taylor Series, its derivation, and its relation to the Binomial Theorem are covered in detail to serve as a basis for the further discussions that we aim to present in this chapter.

To relate natural numbers as a set and the exponential map we need to go into detail and look into the derivation of Taylor Series from the Binomial Theorem.

2 Definitions and Proofs

2.1 Theoretical Background

In this subsection, we will adapt the proof for the following theorem from [3]. This theorem serves as a basis for representing Exponential function as a multiplication of infinitely many, infinitely small steps.

Theorem 2.1.

$$\lim_{n\to inf}\left(1+\frac{x}{n}\right)^n = e^x \tag{3}$$

Proof of Theorem 2.1.

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k} \tag{4}$$

$$= 1 + x + \sum_{k=2}^{n}\frac{n(n-1)\dots(n-(k-1))}{k!}\frac{x^k}{n^k} \tag{5}$$

$$= 1 + x + \frac{x^2}{2!}\left(1-\frac{1}{n}\right) \tag{6}$$

$$+\frac{x^3}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \tag{7}$$

$$\dots$$

$$+\frac{x^n}{n!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{n-1}{n}\right) \tag{8}$$

For n ≥ m, from Theorem 3.19 of [3], the expression can be rewritten as follows.

$$1 + x + \frac{x^2}{2!}\left(1-\frac{1}{n}\right) \tag{9}$$

$$+\frac{x^3}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \tag{10}$$

$$\dots$$

$$+\frac{x^m}{m!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{m-1}{n}\right) \tag{11}$$

as n → inf and m fixed, expression converges to the Taylor Series expansion for e^X,

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^m}{m!} \tag{12}$$

□

To understand the intuition of the relation between the Binomial theorem and the Taylor series we may take a look at the binomial coefficient,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!} \tag{13}$$

where the numerator is the number of k permutations of n objects without repetitions whereas the denominator represents the number of permutations of the k objects. As a result, we get only the k-combinations of the n elements. However, as we multiply the binomial coefficient with the $1/n^k$

as in Equation 4, will cause the number of k-permutations to be selected to 1, a single permutation of k objects.

In the Dyson series, each order of approximation has a number

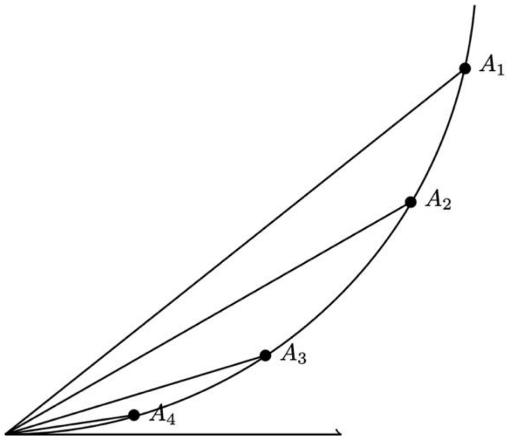

Figure 3. $A_i \in G$ for all i, $A_m \to \infty$ where A_{m-1} / $(1 \ / \ m)$.

of virtual states associated as depicted in a way in Fig. 2. If we consider approximation order as k,

$$U_I^{(k)}(t,t_0) = \sum_{n=1}^{k} \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \ldots \\ \ldots \int_{t_0}^{\tau_{n-1}} V_I(\tau_1)V_I(\tau_2)\ldots V(t_n) \tag{14}$$

Intermediate states for the first two orders of approximation can be written as follows. The operator $J_I^{(k)}$ represents an order denoted by k where $U_I^{(k)}(t,t_0) = \sum_{n=1}^{k} J_I^{(k)}(t,t_0)$.

$$J_I^{(1)}(t,t_0) = \left(-\frac{i}{\hbar}\right)^1 \int_{t_0}^{t} \langle n|V_I(\tau_1)d\tau_1|i\rangle \tag{15}$$

$$J_I^{(2)}(t,t_0) \\ = \left(-\frac{i}{\hbar}\right)^2 \sum_m \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \langle n|V_I(\tau_1)|m\rangle\langle m|V_I(\tau_2)|i\rangle \tag{16}$$

However, this is the case where integration limits are coupled because of the time ordering. Therefore, Equation 17 has interactions time-ordered. On the other hand, if integration limits can be decoupled as follows,

$$U_I^{(k)}(t,t_0) = T\left[\sum_{n=1}^{k} \left(-\frac{i}{\hbar}\right)^n \frac{1}{k!} \int_{t_0}^{t} d\tau_1 \int_{t_0}^{t} d\tau_2 \ldots \\ \ldots \int_{t_0}^{t} V_I(\tau_1)V_I(\tau_2)\ldots V(t_n)\right] \tag{17}$$

where it has an additional term, $1/k!$ corresponds to the ratio of integration region that integral has to cover [5].

2.2 Postulates

Considering X is the generator of the group, where,

$$A(t) = e^{tX} = \lim_{n \to inf} \left(I + \frac{t}{n}X\right)^{\mathrm{n}} \tag{18}$$

Conjecture 2.2.1 (Set Existence).

$$\exists A(A^{\intercal} = A^{-1}) \tag{19}$$

Conjecture postulates that there exist elements of a rotation group with unitary property that can be considered as a set. Therefore, Axiom 0 of ZFC is re-interpreted as a conjecture to be proven to represent rotation group elements as sets.

Corollary 2.2.1. Considering A as a unitary transformation and I as the identity element of a rotation group G.

$$I \equiv \emptyset \tag{20}$$

Proof of Corollary 2.2.1. For all $A \in G$ there is the $I \in G$ where,

$$A = IA \tag{21}$$

For any set V_A,

$$V_A = \emptyset \cup V_A \tag{22}$$

□

Lemma 2.2.1. An infinitely small transformation on a vector $v \in G$ to another element of the group at an infinitely small distance defined as follows,

$$A_\epsilon(v) = \left(I + \frac{t}{n}X\right)v \tag{23}$$

and, correspondingly, $A(s)$ takes s to the next group element as follows,

$$\mathcal{A}(s) = (s \cup s^+) \tag{24}$$

which is the same recurrence relation for von Neumann ordinals, $V_{i+1} = V_i \cup V_i^+$ [6].

$$e^X = \lim_{n \to inf} \underbrace{A_\epsilon(\ldots A_\epsilon(}_{n\ times} v) \ldots) \tag{25}$$

$$\mathcal{A}_s = \lim_{n \to inf} \underbrace{\mathcal{A}(\ldots \mathcal{A}(}_{n\ times} s) \ldots) \tag{26}$$

$$A_\epsilon(v) \equiv \mathcal{A}(s) \tag{27}$$

The term $\mathcal{A}_s$ from Equation 26 will be explained in the Proof of Lemma 2.2.1.

Proof of Lemma 2.2.1.

We will partly take [7]'s introduction to “Generating a path-connected group from a neighborhood of 1” and merge it with Corollary 3.52 of [8] to serve our conclusion.

Since G is path connected, for any $A \in G$ there is a path $A(t)$ in G with $A(0) = 1$ and $A(1) = A$. Any open set that includes I, multiplication by $A(t)$ is a continuous map (with an inverse $A(t)^{-1}$). $A(t)$ is a smooth path in G where the sequence of points $A_m = A(1/m)$ as in Fig. 3.

When considered $\mathcal{O}$ as any open set that includes I,

$$A(t)\mathcal{O} = \{A(t)B : B \in \mathcal{O}\} \tag{28}$$

is an open set that includes $A(t)$. Thus, as t runs from 0 to 1 the open sets $A(t)\mathcal{O}$ cover the image of the path $A(t)$, which is the continuous image of the compact set as t in the range [0,1]. The image of the path lies in a finite union of sets,

$$\underbrace{A(1/m)\mathcal{O}}_{A_{m-1}=A_m A_m^{-1} A_{m-1}} \cup \underbrace{A(1/(m-1))\mathcal{O}}_{A_{m-2}=A_{m-1} A_{m-1}^{-1} A_{m-2}} \cup \ldots \cup A(1)\mathcal{O} \tag{29}$$

therefore, $1 = A_m, A_{m-1}, \ldots, A_1 = A$ can be found on the path $A(t)$. In terms of [7]'s notation with a little modification on our representation,

$$A = A_m A_m^{-1} A_{m-1} A_{m-1}^{-1} \ldots A_1^{-1} A_1 \tag{30}$$

which can be rewritten like [8]'s Lemma 3.48,

$$A = \underbrace{A_m}_{A(0)} \underbrace{A_m^{-1}}_{A(0)^{-1}} \underbrace{A_{m-1}}_{A(1/m)} \underbrace{A_{m-1}^{-1}}_{A(1/m)^{-1}} \ldots \underbrace{A_1^{-1}}_{A((m-1)/m)^{-1}} \underbrace{A_1}_{A(1)} \tag{31}$$

where,

$$A = e^X \tag{32}$$

$$= \underbrace{A(0)}_{I} \underbrace{A(0)^{-1}A(1/m)}_{e^{X_1}} \underbrace{A(1/m)^{-1}A(2/m)}_{e^{X_2}} \ldots$$

$$\ldots \underbrace{A((m-1)/m)^{-1}A(m)}_{e^{X_m}} \tag{33}$$

$$= e^{X_1} e^{X_2} \ldots e^{X_m} \tag{34}$$

which satisfies the decomposition of Equation 25.

For the other half of the proof; Equation 26, we can take a look at Equation 29 and try to decompose Equation 26 in terms of a finite union of terms as shown in Equation 24 which is well-known as von Neumann ordinals,

$$V_0 = \emptyset = \mathcal{A}_0 \tag{35}$$

$$= 0 \tag{36}$$

$$V_1 = \mathcal{A}(\mathcal{A}_0) = \mathcal{A}_1 \tag{37}$$

$$= V_0 \cup V_0^+ = V_0 \cup V_1 \setminus V_0 \tag{38}$$

$$= \{0\} = 1 \tag{39}$$

$$V_2 = \mathcal{A}(\mathcal{A}_1) = \mathcal{A}_2 \tag{40}$$

$$= V_1 \cup V_1^+ = V_1 \cup V_2 \setminus V_1 \tag{41}$$

$$= \{0,1\} = 2 \tag{42}$$

$$\ldots$$

$$V_\alpha = \mathcal{A}(V_{\alpha-1}) = \mathcal{A}_\alpha \tag{43}$$

$$= V_{\alpha-1} \cup V_{\alpha-1}^+ = V_{\alpha-1} \cup V_\alpha \setminus V_{\alpha-1} \tag{44}$$

where V_α is an open set since it does not contain itself.

$$V_\alpha = \bigcup_{i<\alpha} V_i \tag{45}$$

Relation between sets and exponential function derived from the union of open sets. Equations 46 and 47 provide a useful decomposition that shows equivalence.

$$V_\alpha = \underbrace{V_0}_{\underbrace{\underbrace{\emptyset}_{A(1/m)}}_{A_m}} \cup \underbrace{V_1}_{\underbrace{\underbrace{\underbrace{V_0}_{A(1/m)}}_{A_m} \cup \underbrace{\underbrace{V_1 \setminus V_0}_{\mathcal{O}}}_{A_m^{-1}A_{m-1}}}_{A_{m-1}=A\left(\frac{1}{m-1}\right)}} \cup \underbrace{V_2}_{\underbrace{V_1}_{A\left(\frac{1}{m-1}\right)} \cup \underbrace{V_2 \setminus V_1}_{\mathcal{O}}} \cup \ldots \tag{46}$$

$$\ldots \cup \underbrace{V_3}_{A(1/(m-2))\mathcal{O}} \cup \ldots \cup \underbrace{V_{\alpha-2}}_{A(\alpha-2)\mathcal{O}} \cup \underbrace{V_{\alpha-1}}_{A(\alpha-1)\mathcal{O}} \tag{47}$$

where,

$$A_m \equiv \emptyset \tag{48}$$

$$\therefore \mathcal{A}_0 = \emptyset \tag{49}$$

and considering the inverse of $\mathcal{A}$,

$$\mathcal{A}^{-1}(s) = s \setminus \mathcal{A}_1 \tag{50}$$

$$\mathcal{A}^{-i}(s) = s \setminus \mathcal{A}_i \tag{51}$$

$$V_{i+1} = \underbrace{V_i}_{A_i} \cup \underbrace{V_{i+1} \setminus V_i}_{A^{-i}(A_{i+1})} \tag{52}$$

$$= \underbrace{V_i \setminus V_i}_{A^{-i}(A_i)} \cup \underbrace{V_{i+1}}_{A_{i+1}} \tag{53}$$

therefore, Equation 46 can be generalized to

$$\mathcal{A}_\alpha = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_{\alpha-1} \tag{54}$$

where,

$$\mathcal{A}_1 = \underbrace{\mathcal{A}(\mathcal{A}^{-0}(\mathcal{A}_1))}_{\mathcal{A}_1} \tag{55}$$

$$\mathcal{A}_2 = \mathcal{A}^2(\mathcal{A}^{-1}(\mathcal{A}_1)) \tag{56}$$

each $\mathcal{A}_i$ mapped from $\mathcal{A}_j$ through origin by $\mathcal{A}^{-j}(\mathcal{A}_j)$ operation in the same way as $A_{m-1} = A_m A_m^{-1} A_{m-1}$. With that in mind, we can write the equivalent of Equation 30 below,

$$\underbrace{\mathcal{A}^\alpha(\mathcal{A}^{-(\alpha-1)}\underbrace{(\mathcal{A}^{\alpha-1}(\mathcal{A}^{-(\alpha-2)}(\ldots \underbrace{\mathcal{A}^2(\mathcal{A}^{-1}(\underbrace{A^1(A^{-0}(\cdot))}_{e^{X_1}})))}_{e^{X_2}})\ldots))}_{e^{X_{m-1}}})}_{e^{X_m}} \tag{57}$$

Thus,

$$\mathcal{A}(s) = (s \cup s^+) \equiv A_\epsilon(v) = \left(I + \frac{t}{n}X\right) \qquad \square$$

Conjecture 2.2.2.

Depending on the Lemma 2.2.1 derivative operator for $\mathcal{A}_i$ where $i \to \infty$ defined in Equation 58. The derivative operator applies the set difference as $\mathcal{A}_i \backslash s$ and it applies the set difference for each element of the $\mathcal{A}_i \backslash s$ recursively.

$$\frac{d\mathcal{A}_i}{ds} = \left\{ x | \forall y \in (\mathcal{A}_i \backslash s) \; for \; \frac{dy}{ds} \right\} \tag{58}$$

Proof of Conjecture 2.2.2

For a variable X, the derivative of exponential can be reminded as below for comparison,

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots \tag{59}$$

$$\frac{de^X}{dX} = 0 + I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots \tag{60}$$

where $e^X \cong de^X/dX$ and, correspondingly for an order α in von Neumann ordinals derived by $\mathcal{A}_\alpha$ shall provide the same behavior. Given A_α can be differentiated using Equation 58 for $\emptyset$ as follows.

$$A_\alpha = \Big\{ \emptyset \cup \{\emptyset\} \cup \{\emptyset \cup \{\emptyset\}\} \cup \Big\{ \emptyset \cup \{\emptyset\} \cup \big\{\{\emptyset \cup \{\emptyset\}\}\big\} \Big\} \ldots \Big\} \tag{61}$$

$$\frac{dA_\alpha}{d\emptyset} = \{\emptyset \cup \{\emptyset\} \cup \{\emptyset \cup \{\emptyset\}\} \ldots \} \tag{62}$$

where $A_\alpha \cong dA_\alpha/d\emptyset$ where $\alpha \to \infty$.

Proof of Conjecture 2.2.1.

Considering that,

1. Corollary 4.2.1, proves that the empty set is the identity element for any set in the hierarchy.
2. Lemma 2.2.1, to prove that $\mathcal{A}$ corresponds to an infinitely small change.
3. Conjecture 2.2.2, to prove that A_α shows identical behavior as the exponential function when the derivative operator from Equation 58 is applied.

In light of the above findings, we can deduce that a continuous path in a connected matrix Lie group can be represented by the von Neumann ordinals in the extent of proven conjectures.

3 Conclusions

This study introduced formal definitions as a basis for the interpretation of a continuous path in a connected matrix Lie group to be interpreted sets and express mathematically similar properties as Exponential function. This has been made possible by properties of von Neumann ordinals. Derivation of von Neumann ordinals strongly correlates with the derivation of the Taylor series from the Binomial theorem for approximating the Exponential function. Therefore, this introductory work postulated conjectures to interpret the rotations that form a smooth curve in terms of sets, namely von Neumann ordinals.

Introduced concepts relate the empty set to rotation groups which will conceptually allow a rotation group to be represented as a hierarchy of sets, namely von Neumann ordinals. Strangely, the derivative operator introduced in Conjecture 2.2.2 which works on von Neumann ordinals causes a similar effect in comparison to differentiating e. We believe our postulates initiate a research effort toward the discretization of mathematical structures formed by rotations, and eventually the exponential function. Discovering ties of exponential function with a way of interpreting natural numbers may reveal possibilities in terms of combinatorics, Maximum Satisfiability, k-coloring, Evolutionary Algorithms; numerical analysis discretization of integrals; robotics, kinematics and trajectory planning, trajectory generation and cryptography, Diffie-Hellman Key Exchange, Zero-Knowledge proofs, encryption algorithms such as RSA, ElGamal. These applications are only some examples that utilize exponential function.

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5 Author Contributions

In this work Author 1 contributed to forming the idea, literature review, derivation of proofs, evaluation of results, and writing.

6 Ethics Committee Approval and Conflict of Interests

This study does not require ethical committee approval.

This article does not contain any conflict of interest with an organization or person.

7 Reference

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