Pamukkale Üniversitesi Mühendislik Bilimleri Dergisi

Pamukkale University Journal of Engineering Sciences

Vibration analysis of beams with intermediate supports under moving loads

Hareketli yüklere maruz ayaklı kirişlerin titreşiminin incelenmesi

Emre GEMİCİ[1*], Yaşar PALA[1]

[1]Department of Mechanical Engineering, Uludag University, Bursa, Türkiye.
gemici.e@gmail.com, mypala@uludag.edu.tr

Received/Geliş Tarihi: 16.01.2023 Revision/Düzeltme Tarihi: 15.01.2025 doi: 10.5505/pajes.2025.82596
Accepted/Kabul Tarihi: 20.01.2025 Research Article/Araştırma Makalesi

Abstract

In this study, transverse vibration of a beam under a moving singular load and a moving moment is investigated. A simply supported beam with intermediate vertical supports modeled according to Euler-Bernoulli beam theory. The intermediate supports are modeled as consisting of a linear spring and a linear damper. The moving force and the moment, the spring force and the damper force are expressed using Dirac delta functions in the equations of motion. Obtaining the exact solution for this problem with classical methods are quite lengthy and complicated. Beam must be divided into spans between each support. Each span must be solved separately with different set of coordinates having same boundary conditions on support points. As the number of support increases, solution becomes more complicated. However, the present method can be used to solve the problem for the whole beam length without having to separate into various spans regardless of number of supports. Dirac delta functions are converted to series expansions which allows us to get exact solution in form of series expansion. This solution than can be easily calculated by a computer. Dynamic responses of several cases such as various number of supports; different support points; various moving load, moving moment and axial load combinations are examined.

Keywords: Moving load, Moving moment, Simply supported beam, Intermediate supports

Öz

Bu çalışmada, bir kirişin hareketli bir yük ve moment altındaki titreşimi incelenmiştir. Destek ayakları içeren basit mesnetli bir kiriş Euler-Bernoulli kiriş teorisine göre modellenmiştir. Destek ayakları doğrusal bir yay ve doğrusal bir damperden oluştuğu varsayılarak modellenmiştir. Hareketli yük, hareketli moment, yay kuvveti ve damper kuvveti Dirac delta fonksiyonu kullanılarak hareket denklemine dâhil edilmiştir. Klasik yöntemler ile bu problemin kesin çözümünü elde etmek oldukça uzun ve karmaşıktır. Her bir ayak için kiriş, ayaklar arası parçalara bölünerek ele alınmalıdır. Bu parçaların her biri, sınır şartları destek ayak noktalarında olacak şekilde ayrı koordinat eksenlerinde çözümlenmelidir. Ayak sayısı arttıkça çözüm daha da zorlaşacaktır. Mevcut çalışmaya konu yöntemle ayak sayısından bağımsız olarak, kirişi parçalara ayırmadan tüm kiriş boyunca tek bir koordinat ekseninde çözüm elde etmek mümkündür. Dirac delta fonksiyonu seri açılımlarına dönüştürülerek seri açılımı halinde kesin çözüme elde edilmiştir. Çözüm, basit bir bilgisayar programı yazarak hesaplanabilir. Bu çözümle kirişin farklı ayak sayıları; farklı konumlarındaki ayaklar; farklı hareketli yük, hareketli moment ve eksenel yük gibi çeşitli durumlardaki dinamik davranışı incelenmiştir.

Anahtar kelimeler: Hareketli yük, Hareketli moment, Basit mesnetli kiriş, Destek ayakları

1 Introduction

Vibration analysis of beams under moving loads can be used to model and study the dynamics of a bridge traveled by a vehicle. Moving load problems and beam with intermediate elastic or viscous supports have been studied by various researchers so far. Frýba [1] solved the moving load problem on a simply supported beam with integral transformation methods in his book dedicated to vibrations under moving loads. However, this method is not suitable when intermediate supports are present in the system. Kameswara [2] used two span beam approach to obtained frequency and mode shapes expressions of a clamped-clamped beam with intermediate elastic support. Lee [3] analyzed the transverse vibration of a beam with intermediate point constraints subject to a moving load. He used the energy methods to get the governing equation and solved it with assumed mode method. Esmailzadeh and Ghorashi [4] solved vibration of a Timoshenko beam subjected to moving distributed mass by finite difference method. Reis et al. [5] investigated a bridge with many vertical support under moving load. They expressed the moving load and the spring and damping loads of the supports as distributed functions. Obtaining an exact solution is greatly simplified with this method. Uzzal et al. [6] found an analytical solution for a beam subjected to moving load supported by Pasternak foundation using Fourier transform technique. Senalp et al. [7] utilized Galerkin method and achieved to obtain a solution for a beam on linear and nonlinear Pasternak foundation subjected to moving force. These two solutions [6, 7] are valid for a distributed Pasternak foundation model and not applicable when independent vertical supports are present. Zhang and Shepard [8] developed a shape function configuration mothed for a Euler-Bernoulli beam with two intermediate supports excited by moving pressure wave loads. Chawda and Murugan [9] studied a cantilever beam under combined moving moment, torque and load. They obtained the governing equations using energy methods and then solved using Laplace transform. Luo et al. [10] obtained closed-form solutions for free vibration of beams with intermediate supports using the generalized function method.

It can be seen that some researchers succeeded to obtain closed form solutions for free vibration of a beam with intermediate supports or constraints. Also some researchers used numerical methods to solve moving load problems when supports are included in the equation of motion. In this study analytical solution for a simply supported beam including intermediate supports has been formulated. Euler-Bernoulli beam theory was used while formulating the equation of motion. Defining

*Corresponding author/Yazışılan Yazar

intermediate supports by Dirac delta functions and converting them to their Fourier series expansions in the equation of motion made it possible to get an ordinary differential equation that can be solved easily.

2 Equation of motion

Consider an axially loaded simply supported beam under a moving load F_0, a moving moment M_0 shown in Fig. 1. The beam is supported by a linear spring of spring constant k and a linear damper of damping coefficient c. The speed of the moving force is of a constant value v_0 and expressed using Dirac delta function such that

$$F(x,t) = F_0 \delta(x - v_0 t) \tag{1}$$

The speed of the moving moment has a constant value of v_1 and is similarly expressed as

$$M(x,t) = M_0 \frac{\partial \delta(x - v_1 t)}{\partial x} \tag{2}$$

Transverse deflection of the beam at point x is denoted as u, time is denoted as t and elastic and viscous supports are assumed to be located at points $x=s_3$ and $x=s_4$ respectively. Then the equation of motion for forced transverse vibration of a uniform Euler-Bernoulli beam [11] can be written as

$$EI \frac{\partial^4 u}{\partial x^4} - P \frac{\partial^2 u}{\partial x^2} + \rho A \frac{\partial^2 u}{\partial t^2} = F_0 \delta(x - v_0 t) + M_0 \frac{\partial \delta(x - v_1 t)}{\partial x} - ku\delta(x - s_3) - c\frac{\partial u}{\partial t}\delta(x - s_4) \tag{3}$$

where E is the Young's modulus, I is the cross-sectional moment of inertia of the beam, ρ is the mass density and A is the cross-sectional area of the beam. ρ and A are assumed to be constant.

It is assumed that total solution of beam vibration is the sum of the normal modes which are also known as mode superposition method. Then, the deflection can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)\, T_n(t) \tag{4}$$

The simply supported beam has the following mode shape functions:

$$X_n(x) = \sin(\omega_n x) \tag{5}$$

where

$$\omega_n = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots \tag{6}$$

Then, the solution of the simply supported beam takes the form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(\omega_n x) \tag{7}$$

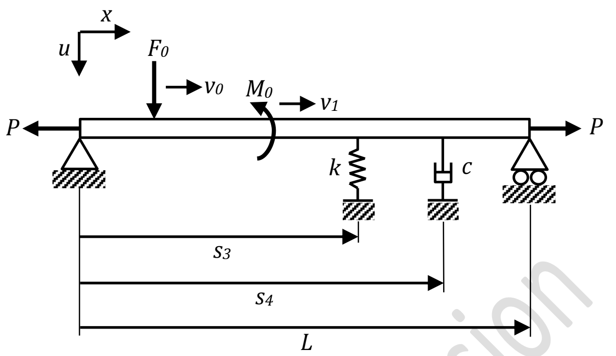

Figure 1. Simply supported beam with intermediate supports under moving load and moment.

Substituting Eq. (7) into Eq. (3) yields

$$\begin{aligned} &EI \sum_{n=1}^{\infty} \omega_n^4\, T_n(t)\sin(\omega_n x) + P \sum_{n=1}^{\infty} \omega_n^2\, T_n(t)\sin(\omega_n x) \\ &+ \rho A \sum_{n=1}^{\infty} \frac{\partial^2 T_n(t)}{\partial t^2}\sin(\omega_n x) \\ &= F_0\delta(x - v_0 t) + M_0 \frac{\partial \delta(x - v_1 t)}{\partial x} \\ &- k\delta(x - s_3) \sum_{n=1}^{\infty} T_n(t)\sin(\omega_n x) \\ &- c\delta(x - s_4) \sum_{n=1}^{\infty} \frac{\partial T_n(t)}{\partial t}\sin(\omega_n x) \end{aligned} \tag{8}$$

The Force F_0 in Eq. (8) can be expanded into Fourier sine series as

$$F_0\delta(x - v_0 t) = \frac{2F_0}{L}\sum_{n=0}^{\infty} \sin(\omega_n v_0 t)\sin(\omega_n x) \tag{9}$$

In the same manner, the moment M_0 can be written in series form as

$$M_0 \frac{\partial \delta(x - v_1 t)}{\partial x} = -\frac{2M_0}{L}\sum_{n=1}^{\infty} \omega_n \cos(\omega_n v_1 t)\sin(\omega_n x) \tag{10}$$

The supports can also be written in series form as

$$\begin{aligned} &k\delta(x - s_3)\sum_{n=1}^{\infty} T_n(t)\sin(\omega_n x) \\ &= \frac{2k}{L}\sum_{n=1}^{\infty} T_n(t)\sin^2(\omega_n s_3)\sin(\omega_n x) \end{aligned} \tag{11}$$

and

$$\begin{aligned} &c\delta(x - s_4)\sum_{n=1}^{\infty} \frac{\partial T_n(t)}{\partial t}\sin(\omega_n x) \\ &= \frac{2c}{L}\sum_{n=1}^{\infty} \frac{\partial T_n(t)}{\partial t}\sin^2(\omega_n s_4)\sin(\omega_n x) \end{aligned} \tag{12}$$

Substituting Eqs. (9), (10), (11), and (12) into Eq. (8) yields following form of the equation:

$$
\begin{aligned}
&EI\sum_{n=1}^{\infty}\omega_n^4\,T_n(t)\sin(\omega_n x)+P\sum_{n=1}^{\infty}\omega_n^2\,T_n(t)\sin(\omega_n x)\\
&+\rho A\sum_{n=1}^{\infty}\frac{\partial^2 T_n(t)}{\partial t^2}\sin(\omega_n x)\\
&=\frac{2F_0}{L}\sum_{n=1}^{\infty}\sin(\omega_n v_0 t)\sin(\omega_n x)\\
&-\frac{2M_0}{L}\sum_{n=1}^{\infty}\omega_n\cos(\omega_n v_1 t)\sin(\omega_n x)\\
&-\frac{2k}{L}\sum_{n=1}^{\infty}T_n(t)\sin^2(\omega_n s_3)\sin(\omega_n x)\\
&-\frac{2c}{L}\sum_{n=1}^{\infty}\frac{\partial T_n(t)}{\partial t}\sin^2(\omega_n s_4)\sin(\omega_n x)
\end{aligned}
\tag{13}
$$

It is obvious that term $\sin(\omega_n x)$ on both sides of Eq. (13) can be cancelled out. Since the equation must be satisfied for each n, Eq. (13) reads

$$
\begin{aligned}
&EI\omega_n^4\,T_n(t)+P\omega_n^2\,T_n(t)+\rho A\ddot{T}_n(t)\\
&=\frac{2F_0}{L}\sin(\omega_n v_0 t)-\frac{2M_0}{L}\omega_n\cos(\omega_n v_1 t)\\
&-\frac{2k}{L}T_n(t)\sin^2(\omega_n s_3)-\frac{2c}{L}\dot{T}_n(t)\sin^2(\omega_n s_4)
\end{aligned}
\tag{14}
$$

The rearrangement of Eq. (14) gives the following equation:

$$
\begin{aligned}
&\ddot{T}_n(t)+a_n\dot{T}_n(t)+b_n T_n(t)\\
&=\frac{2}{\rho AL}\left[F_0\sin(\omega_n v_0 t)-M_0\omega_n\cos(\omega_n v_1 t)\right]
\end{aligned}
\tag{15}
$$

where

$$
a_n=\frac{2c}{\rho AL}\sin^2(\omega_n s_4)
\tag{16}
$$

and

$$
b_n=\frac{1}{\rho A}\left(EI\omega_n^4+P\omega_n^2+\frac{2k}{L}\sin^2(\omega_n s_3)\right)
\tag{17}
$$

Assuming that the beam is initially at rest, the initial conditions are written in the forms

$$
T_n(0)=\dot{T}_n(0)=0
\tag{18}
$$

Using the initial conditions in Eq. (18), Eq. (15) can be solved [12] and $T_n(t)$ can finally be obtained.

3 Numerical examples

A beam with properties listed in Table 1 is investigated under various load and support configurations. The summation of Eq. (7) is calculated by a computer program for n=100 terms and the beam shape $u(x,t)$ is obtained. The maximum deflection of the beam in dynamic response was compared to maximum static deflection when loaded at the midpoint with the same magnitude of the moving load. The beam is assumed to be initially at rest.

3.1 Various Cases for the Beam

The static midpoint deflection of the beam when 20000 N is applied at $L/2$ calculated as 0.238 m from deflection formula $u_s=FL^3/48EI$ [13] and this value can be used to verify the formulation method of this work. In Fig. 2, the response of the beam at different times are plotted in the case that only moving force is acting on the beam. Similarly, in Fig. 3, the response of the beam is shown in the case where only the moving moment acts. In Fig. 4 the effect of axial load on the response of the beam in the case where the moving force acts are depicted. In Figs. 6-9, the responses of the beam under moving load are shown for various support configurations assuming each support has both elastic and viscous characteristics.

Table 1. Properties of the beam

Description	Symbol	Value	Unit
Beam width		0.1	m
Beam height		0.1	m
Beam length	L	10	m
Cross-sectional area	A	0.01	m²
Area moment of inertia	I	8.33e-6	m⁴
Density	ρ	7800	kg/m³
Modulus of elasticity	E	2.1e+11	N/m²

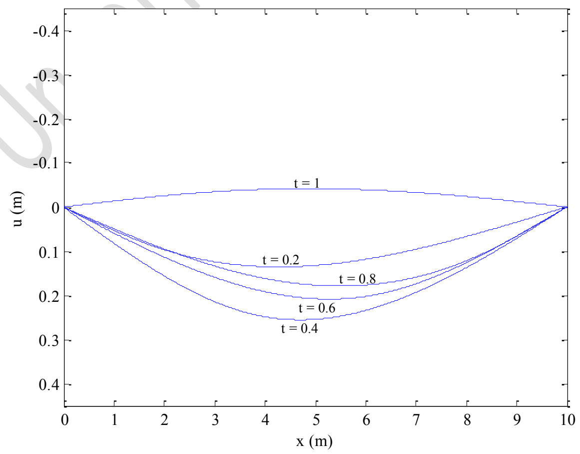

Figure 2. Shapes of the beam under moving load only, F_0=20000 N, v_0=10 m/s, M_0=0, P=0.

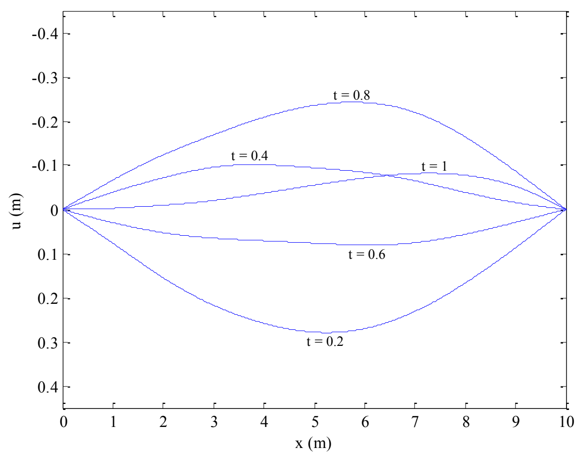

Figure 3. Shapes of the beam under moving moment only, F_0=0, M_0=40000 N.m, v_1=10 m/s, P=0.

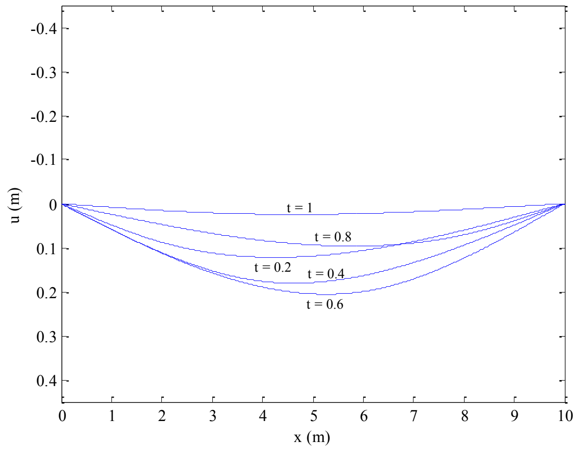

Figure 4. Shapes of the beam under moving load and axial tension force, F_0=0, M_0=0, P=40000 N.

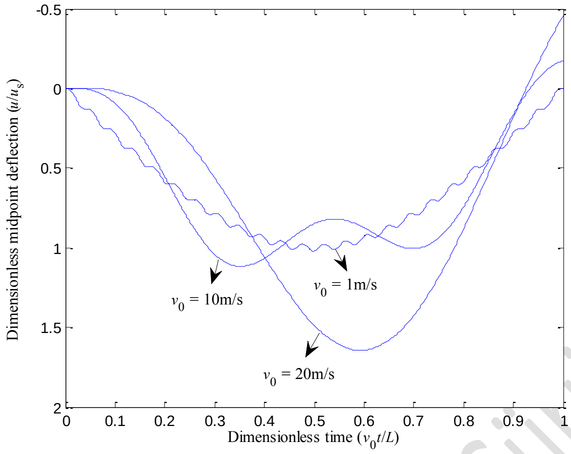

Figure 5. Dimensionless midpoint deflection under the moving loads, v_0=1 m/s, 10 m/s and 20 m/s.

When the magnitude of the force changes, it is noticed that frequency of the beam vibration doesn't change. Deflection changes directly proportional to the magnitude of the force. Also when change of force's speed is investigated, it is seen that when speed increases, the maximum deflection of the beam also increases. At lower speeds, there are more beam oscillations than higher speeds. This is shown in the Fig. 5 as midpoint deflection vs time plot on dimensionless coordinates where time coordinate is calculated as $v_0 t$/L and u coordinate is calculated as u/u_s. Three different magnitudes of forces plotted as F_0=200 N, F_0=2000 N and F_0=20000 N with three different speeds. Total nine plots are obtained. It is observed that all forces with same speed have same plot as dimensionless deflection is independent from the force as stated before. Also increase in deflection as speed increases can be clearly observed.

3.2 Comparison of The Results with FEA

A finite element analysis has been done using Ansys software to compare the results of this method with those results of the finite element analysis software. Two transient structural analysis were performed with the same properties of the beam as previous numerical examples and its model can be seen in Fig. 10. Material is chosen as structural steel. The beam was divided into 100 elements each having 0.1 m length. Bottom edge of the beam on the left hand side was fixed. Bottom edge on the right hand side was fixed in X and Z directions and left free in Y direction in order to model the beam supports. The analysis was divided into 21 steps and in each step the force

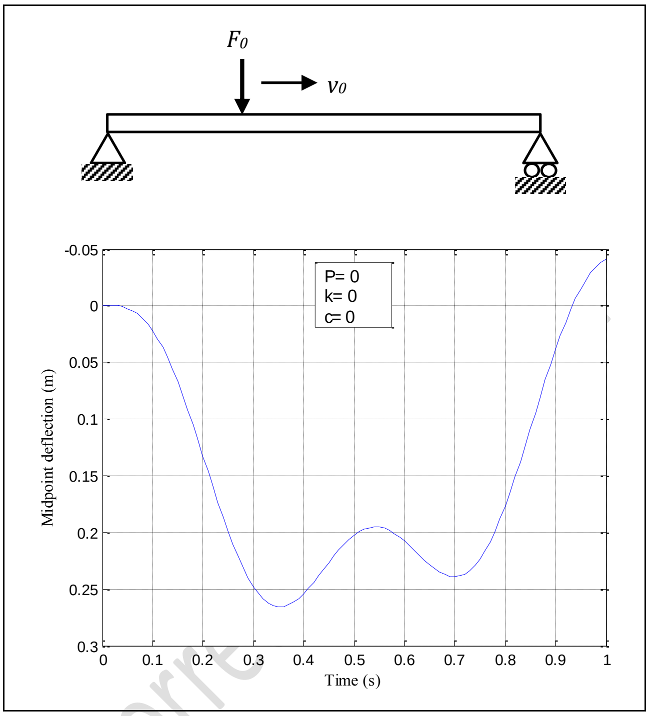

Figure 6. Midpoint deflection with no supports, F_0=20000 N, v_0=10 m/s, M_0=0, P=0.

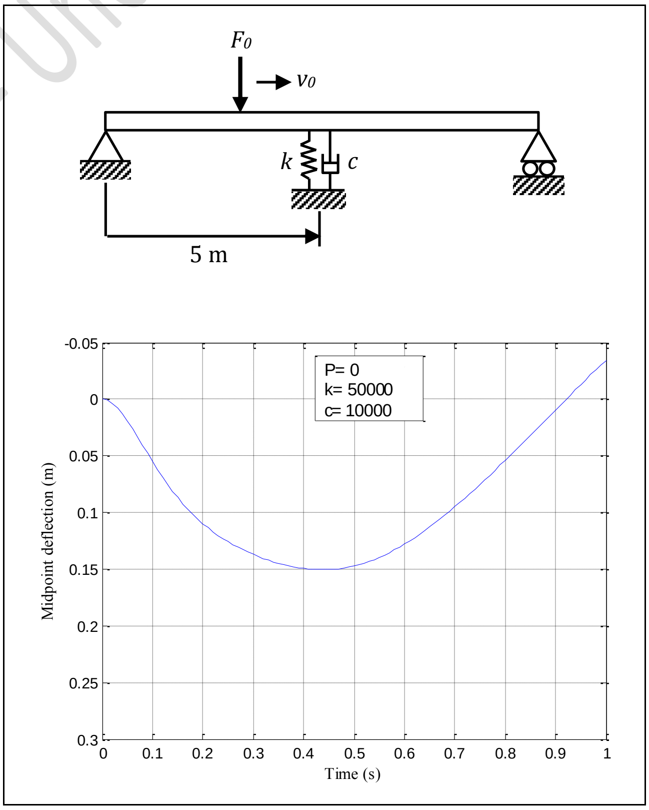

Figure 7. Midpoint deflection with support at x=5, F_0=20000 N, v_0=10 m/s, M_0=0, P=0.

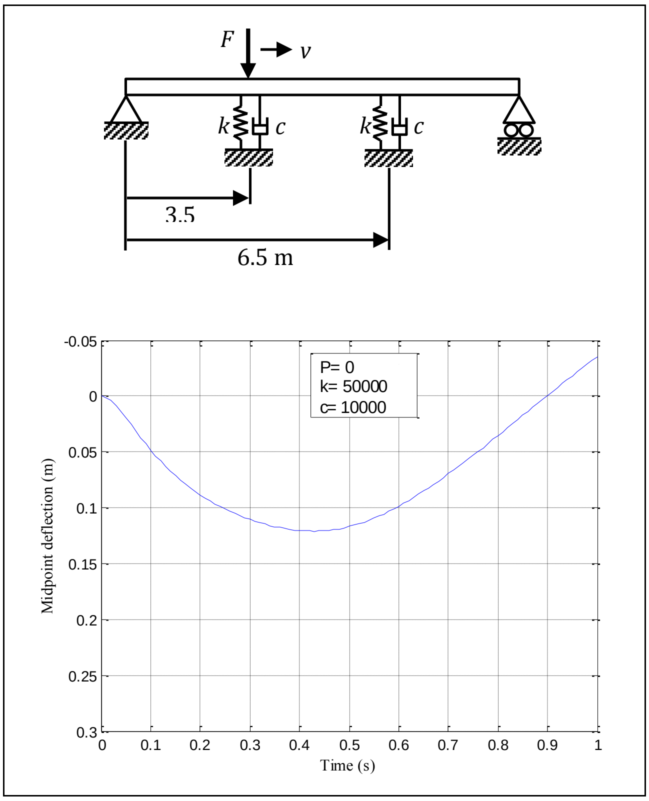

Figure 8. Midpoint deflection with two supports at x=3.5 and x=6, F_0=20000 N, v_0=10 m/s, M_0=0, P=0.

was applied on element borders with 0.5 m distance. In first analysis, the deflections of the beam subjected to same magnitude of moving load have been obtained. The midpoint deflection versus time diagram is plotted in Fig. 11. When comparing with present method's deflection, it can be clearly seen that Ansys results are in good agreement with the present method.

The midpoint deflection of the present method in the case of moving moment shown in Fig. 12 is also compared with Ansys results. It is obvious that both results are again in good agreement especially considering their frequencies.

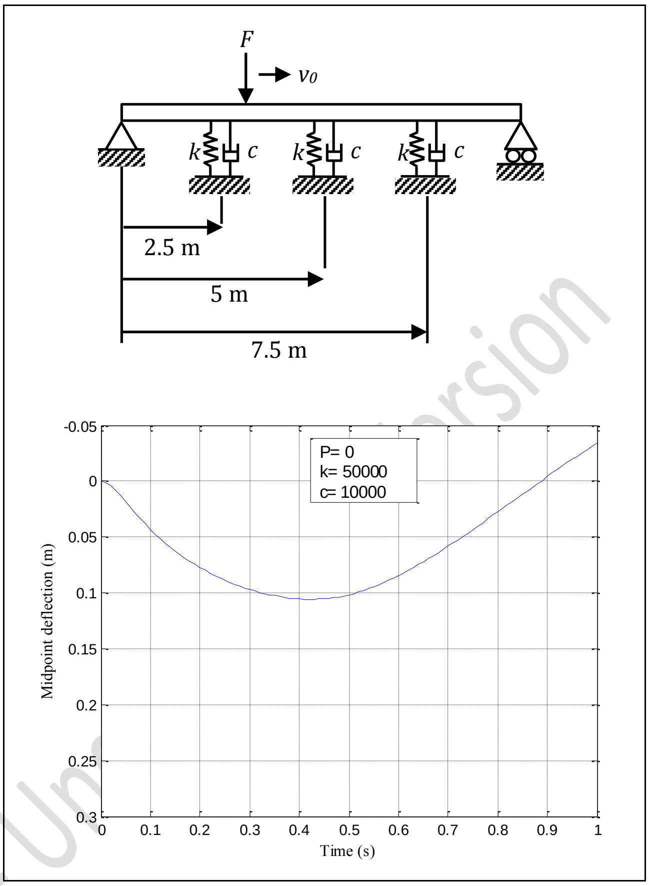

Figure 9. Midpoint deflection with three supports at x=2.5, x=5 and x=7.5, F_0=20000 N, v_0=10 m/s, M_0=0, P=0.

4 Conclusion

In this study, a method for solving equation of motion of a uniform Euler-Bernoulli beam under moving load was developed. This method can be used to predict dynamic responses of bridges traveled by a vehicle. The maximum static

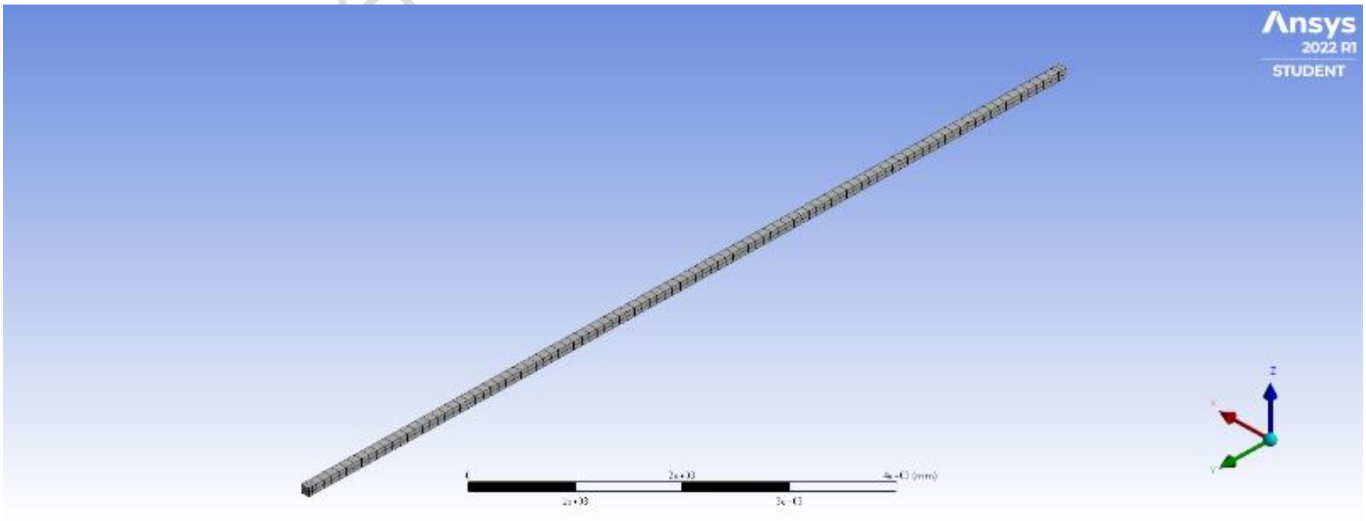

Figure 10. Ansys finite element model.

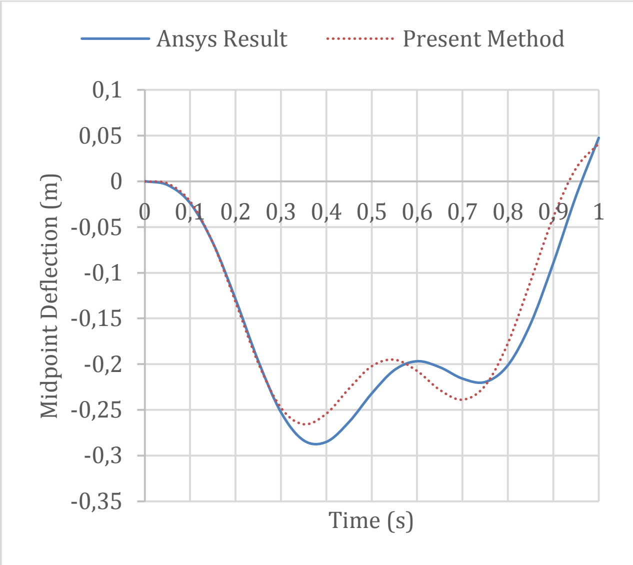

Figure 11. Midpoint deflection of Ansys results under moving load, *F*=20000 N

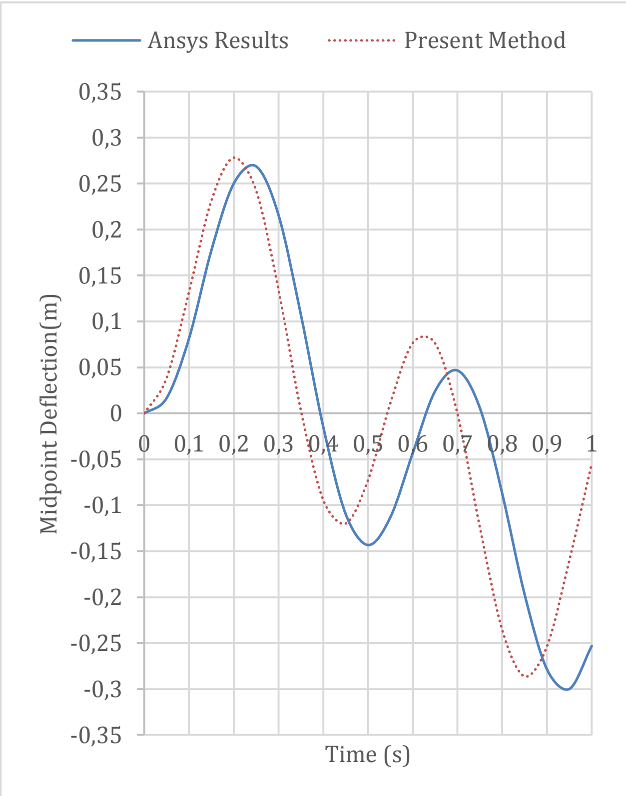

Figure 12. Midpoint deflection of Ansys results under moving moment, *M*=40000 N.m

deflection at the midpoint of the beam is close to the maximum deflection under a moving load with the same magnitude. This helps us to validate the accuracy of the method. When the axial load is applied on the beam, maximum deflection decreases as expected. Thus the present method can be used for structural elements under tensional loads. The most remarkable aspect of the method is that it allows implementing supports into the equation and makes it possible to have an exact solution rather than numerical methods. Also multiple moving loads or moving

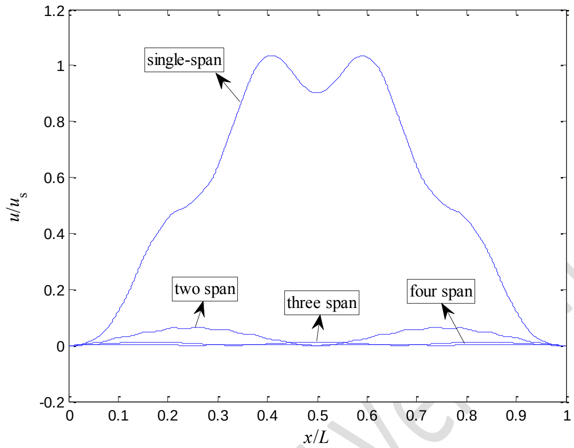

Figure 13. Deflection under the moving load on dimensionless coordinates at v_0=17.3 m/s for single-span, two span, three span and four span beams

moments at different speeds and supports at different locations can be implemented into the equation and can be solved with little effort. In case of multiple loads, it must be noted that the solution will only cover the time interval when all the loads are acting on the beam. In case of multiple supports, the method allows to compare different support configurations and can be used for the optimization of supports under bridges. For example, in Figs. 7-9, it can be seen that the maximum deflection with two supports is less than the maximum deflection with one support. But, there isn't a significant difference in maximum deflections of two and three supports. Thus, the example with two supports can be considered more effective than that of the beam with three supports.

In Fig. 12, dimensionless deflection under the moving load for different intermediate support settings are given. This is a comparison of the present method with method put forward by Lee [3] in his study. *k* is taken as a very large value (10^{12} N/m) so that supports behave like fixed supports. This way the beam is divided into spans and its dynamic behavior is investigated for different span cases. v_0 is taken as 17.3 m/s. The beam properties also taken as the same properties specified in reference [2]. Results are found to be in very good agreement with the results plotted in reference [2].

Results of the present method are also compared with results of finite element analysis. Ansys software was used and results were in good agreement with present method. Slight differences are possibly due to present method's Euler-Bernoulli beam theory approach. Finite element analysis software takes into consideration displacement in all directions whereas Euler-Bernoulli beam theory assumes that displacements are only transversal. Regardless the method used to define a moving load in a finite element analysis software can be used for different applications where a moving load needs to be defined as a boundary condition.

5 Acknowledgment

6 Author contribution statements

Author 1 contributed to this study by the formation of the idea, the literature review, performing analyzes and writing. Author 2 contributed by the formation of the idea and critical review.

7 Ethics committee approval and conflict of interest statement

There is no need to obtain permission from the ethics committee for the article prepared.

There is no conflict of interest with any person / institution in the article prepared.

8 References

[1] Frýba L. *Vibration of solids and structures under moving loads.* Groningen, Netherlands, Noordhoff International Publishing, 1972.

[2] Kameswara R.C. "Frequency analysis of clamped-clamped uniform beams with intermediate elastic support". *Journal of Sound and Vibration*, 133(3), 502-509, 1989.

[3] Lee HP. "Dynamic response of a beam with intermediate point constraints subject to a moving load". *Journal of Sound and Vibration*, 171(3), 361-368, 1994.

[4] Esmailzadeh E, Ghorashi M. "Vibration analysis of a Timoshenko beam subjected to a moving mass". *Journal of Sound and Vibration*, 199(4), 615-628, 1997.

[5] Reis M, Pala Y, Karadere G. "Dynamic analysis of a bridge supported with many vertical supports under moving load". *The Baltic Journal of Road and Bridge Engineering*, 3(1), 14-20, 2008.

[6] Uzzal RUA, Bhat RB, Ahmed W. "Dynamic response of a beam subjected to moving load and moving mass supported by Pasternak foundation". *Shock and Vibration*, 19, 205–220, 2012.

[7] Senalp AD, Arikoglu A, Ozkol I, Dogan VZ. "Dynamic response of a finite length Euler-Bernoulli beam on linear and nonlinear viscoelastic foundations to a concentrated moving force". *Journal of Mechanical Science and Technology*, 24 (10), 1957-1961, 2010.

[8] Zhang B, Shepard S. "Dynamic responses of supported beams with intermediate supports under moving loads". *Shock and Vibration*, 19, 1403–1413, 2012.

[9] Chawda D, Murugan S. "Dynamic response of a cantilevered beam under combined moving moment, torque and force". *International Journal of Structural Stability and Dynamics*, 2050065, 2020.

[10] Luo J, Zhu S, Zhai W. "Exact closed-form solution for free vibration of Euler-Bernoulli and Timoshenko beams with intermediate elastic supports". *International Journal of Mechanical Sciences* 213, 106842, 2022.

[11] Rao SS. *Mechanical Vibrations.* 6th ed. Malaysia, Pearson Prentice Hall, 2018.

[12] Kreyszig E, *Advanced Engineering Mathematics,* Jefferson City, MO, RR Donnelley & Sons Company, 2011.

[13] Case J, Chilver AH, Ross CTF. *Strength of Materials and Structures.* 4th ed. London, England, John Wiley & Sons Inc., 1999