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Free Vibration Analysis of the Axial-Loaded Timoshenko Multiple-Step Beam Carrying Multiple Elastic-Supported Rigid Bars

Çok Sayıda Elastik Mesnetli Rijit Çubuk Taşıyan Eksenel Yüklü Çok Kademeli Timoshenko Kirişinin Serbest Titreşim Analizi

Yusuf YESILCE

Dokuz Eylul University, Engineering Faculty, Civil Engineering Department, 35160, Buca, Izmir

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ABSTRACT

In this paper, the natural frequencies of the axial-loaded Timoshenko multiple-step beam carrying multiple elastic-supported rigid bars are calculated. At first, the coefficient matrices for the elastic-supported rigid bars, the step change in cross-section, left-end support and right-end support of the multiple-step beam are derived. Next, the numerical assembly technique is used to establish the overall coefficient matrix for the whole vibrating system. Finally, equating the overall coefficient matrix to zero one determines the natural frequencies of the system. The natural frequencies of the beams by using secant method for the different values of axial force are presented in tables.

Keywords: *Axial force effect, Free vibration, Multiple elastic-supported rigid bars, Numerical assembly technique, Timoshenko multiple-step beam.*

ÖZET

Bu çalışmada, çok sayıda elastik mesnetli rijit çubuk taşıyan, eksenel yüke maruz, çok kademeli Timoshenko kirişinin doğal frekansları hesaplanmıştır. İlk olarak, elastik mesnetli rijit çubukların, kiriş en kesitinin değiştiği noktaların, sol uç mesnetin ve sağ uç mesnetin kat sayılar matrisi elde edilmiştir. Sonra, nümerik toplama tekniği kullanılarak titreşen sistemin bileşik katsayılar matrisi kurulmuştur. Son olarak, bileşik katsayılar matrisinin determinantı sıfıra eşitlenerek sistemin doğal frekansları hesaplanmıştır. Farklı eksenel kuvvet değerleri için secant metodu kullanılarak hesaplanan kiriş doğal frekans değerleri tablolar halinde sunulmuştur.

Anahtar Kelimeler: *Eksenel kuvvet etkisi, Serbest titreşim, Çok sayıdaki elastik mesnetli rijit çubuklar, Nümerik toplama tekniği, Çok kademeli Timoshenko kirişi.*

1. INTRODUCTION

Beams with step changes in cross-section occur in civil and mechanical engineering structural elements. The free vibration characteristics of a uniform or non-uniform beam carrying various concentrated elements (such as intermediate point masses, rotary inertias, linear springs, rotational springs, etc.) is an important problem in engineering. Thus, a lot of studies have been published in this area.

The normal mode summation technique to determine the fundamental frequency of the cantilever beams carrying masses and springs was used by Gürgöze, (1984;1985). Hamdan and Jubran investigated the free and forced vibrations of a restrained uniform beam carrying an intermediate lumped mass and a rotary inertia (Hamdan and Jubran, 1991). Zhou investigated the free vibration analysis of a cantilever beam carrying a heavy tip mass by a translational spring and a rotational spring (Zhou, 1997). Gürgöze et al. solved the eigenfrequencies of a cantilever beam with attached tip mass and a spring-mass system and studied the effect of an attached spring-mass system on the frequency spectrum of a cantilever beam

* Yazışılan yazar/Corresponding author. E-posta adresi/E-mail address : yusuf.yesilce@deu.edu.tr (Y. Yeşilce)

(Gürgöze, 1996; Gürgöze and Batan, 1996; Gürgöze, 1998). Moreover, they studied on two alternative formulations of the frequency equation of a Bernoulli-Euler beam to which several spring-mass systems being attached in-span and then solved for the eigenfrequencies. Liu et al. formulated the frequency equation for beams carrying intermediate concentrated masses by using the Laplace Transformation Technique (Liu et al., 1998). Wu and Chou obtained the exact solution of the natural frequency values and mode shapes for a beam carrying any number of spring masses (Wu and Chou, 1999). The free vibration analysis of a uniform Timoshenko beam carrying multiple spring-mass systems was studied by Wu and Chen (2001). Gürgöze and Erol investigated the forced vibration responses of a cantilever beam with single intermediate support (Gürgöze and Erol, 2001; 2002). Naguleswaran investigated the natural frequencies and mode shapes of a Bernoulli-Euler beam with one-step change in cross-section and with ends on classical supports by equating the second order determinant to zero (Naguleswaran, 2002a). In the other study, Naguleswaran obtained the natural frequencies and mode shapes of a Bernoulli-Euler beam on elastic end supports and with up to three-step changes in cross-sections by equating the fourth order determinant to zero (Naguleswaran, 2002b). Chen and Wu obtained the exact natural frequencies and mode shapes of the non-uniform beams with multiple spring-mass systems (Chen and Wu, 2002). Naguleswaran obtained the natural frequency values of the beams on up to five resilient supports including ends and carrying several particles by using Bernoulli-Euler Beam Theory (BET) and a fourth-order determinant equated to zero (Naguleswaran, 2002c, 2003a). Chen investigated the natural frequencies and mode shapes of the non-uniform beams carrying multiple various concentrated elements (Chen, 2003). The vibration and stability of an axial-loaded Bernoulli-Euler beam with step changes in cross-sections was investigated by Naguleswaran (2003b; 2004a). In the other study, Naguleswaran investigated the vibration of an axial-loaded Bernoulli-Euler beam carrying a non-symmetrical rigid body at the step (Naguleswaran, 2004b). Lin and Chang studied the free vibration analysis of a multi-span Timoshenko beam with an arbitrary number of flexible constraints by considering the compatibility requirements on each constraint point and using a transfer matrix method (Lin and Chang, 2005). Lin and Tsai determined the exact natural frequencies together with the associated mode shapes for Bernoulli-Euler multi-span beam carrying multiple point masses (Lin and Tsai, 2005). Koplow et al. studied the closed form solutions for the dynamic analysis of Bernoulli-Euler beams with step changes in cross-sections (Koplow et al., 2006). In the other study, Lin and Tsai investigated the free vibration characteristics of Bernoulli-Euler multiple-step beam carrying a number of intermediate lumped masses and rotary inertias (Lin and Tsai, 2006). The natural frequencies and mode shapes of Bernoulli-Euler multi-span beam carrying multiple spring-mass systems were determined by Lin and Tsai (2007). Wang et al. studied the natural frequencies and mode shapes of a uniform Timoshenko beam carrying multiple intermediate spring-mass systems with the effects of shear deformation and rotary inertia (Wang et al., 2007). Wu and Chen investigated the free vibration analysis of a non-uniform Bernoulli-Euler beam with various boundary conditions and carrying multiple concentrated elements by using continuous-mass transfer matrix method (Wu and Chen, 2008). Yesilce et al. investigated the effects of attached spring-mass systems on the free vibration characteristics of the 1-4 span Timoshenko beams (Yesilce et al., 2008). In the other study, Yesilce and Demirdag described the determination of the natural frequencies of vibration of Timoshenko multi-span beam carrying multiple spring-mass systems with axial force effect (Yesilce and Demirdag, 2008). Lin investigated the free and forced vibration characteristics of Bernoulli-Euler multi-span beam carrying a number of various concentrated elements (Lin, 2008). Yesilce investigated the effect of axial force on the free vibration of Reddy-Bickford multi-span beam carrying multiple spring-mass systems (Yesilce, 2010). Lin investigated the free vibration characteristics of non-uniform Bernoulli-Euler beam carrying multiple elastic-supported rigid bars (Lin, 2010).

Multiple-step beams carrying multiple elastic-supported rigid bars are widely used in engineering applications, but in the literature for free vibration analysis of such structural systems; Bernoulli-Euler Beam Theory

(BET) without axial-force effect is used. The literature regarding the free vibration analysis of Bernoulli-Euler single-span beams carrying a number of spring-mass systems, Bernoulli-Euler multiple-step and multi-span beams carrying multiple spring-mass systems and multiple point masses are plenty, but that of Timoshenko multiple-step beams carrying multiple elastic-supported rigid bars with axial force effect is fewer. The purpose of this paper is to utilize the numerical assembly technique to determine the exact natural frequencies of the axial-loaded Timoshenko multiple-step beam carrying multiple elastic-supported rigid bars. The model allows analyzing the influence of the shear and axial force effects, intermediate elastic-supported rigid bars on the free vibration analysis of the multiple-step beams by using Timoshenko Beam Theory (TBT). In this paper, the exact natural frequencies of the beams are calculated and the effects of the axial force and the influence of the shear are investigated by using the computer package, Matlab. Unfortunately, a suitable example that studies the free vibration analysis of Timoshenko multiple-step beam carrying multiple elastic-supported rigid bars with axial force effect has not been investigated by any of the studies in open literature so far.

2. THE MATHEMATICAL MODEL AND FORMULATION

An axial-loaded Timoshenko beam supported by s pins by including those at the two ends of the beam with k-step changes in cross-sections and carrying n elastic-supported rigid bars is presented in Figure1. From Figure 1, the total number of stations is. $M' = k + n + s$.The kinds of coordinates which are used in this study are given below:

$x_{v'}$ are the position vectors for the stations, $(1 \le v' \le M')$,

x_p^* are the position vectors of the elastic-supported rigid bars, $(1 \le p \le n)$,

$\bar{x}_r$ are the position vectors of the step changes in cross-sections, $(1 \le r \le k)$

$\hat{x}_j$ are the position vectors of the supports, $(1 \le j \le s)$.

From Figure 1, the symbols of $1', 2', \ldots, v', \ldots, M' - 1, M'$ above the x-axis refer to the numbering of stations. The symbols of 1, *2, …,p, …,n* below the x-axis refer to the numbering of the elastic-supported rigid bars. The symbols of (1), (*2*), ...,(*r*), ...,(*k*) below the x-axis refer to the numbering of the step changes in cross-sections. The symbols of $[1], [2], \ldots, [j], \ldots, [s]$ below the x-axis refer to the numbering of the supports.

In Figure 1, the each elastic-supported rigid bar is fixed on the beam and possessing its own mass mp and rotary inertia $I_{0,p}$ and supported by a translational spring R_p and a rotational spring J_p.

Each of the symbols "×" denotes the fixed point of an elastic-supported rigid bar with the beam and each of the symbols "." denotes the center of gravity of the rigid bar. In Figure1, $d_{m,p}$ is the distance between the fixed point of the elastic-supported rigid bar and its center of gravity and $d_{k,p}$ is the distance between the fixed point and the translational and the rotational springs supporting the rigid body.

Using Hamilton's principle, the equations of motion for the axial-loaded Timoshenko multiple-step beam can be written as:

$$EI_i \cdot \frac{\partial^2 \theta_i(x_i,t)}{\partial x_i^2} + \frac{GA_i \cdot}{\bar{k}} \cdot \left(\frac{\partial y_i(x_i,t)}{\partial x_i} - \theta_i(x_i,t) \right) - \bar{m}_i \cdot \frac{\partial^2 \theta_i(x_i,t)}{\partial t^2} = 0 \tag{1.a}$$

$$\frac{GA_i}{\bar{k}} \cdot \left(\frac{\partial^2 y_i(x_i,t)}{\partial x_i^2} - \frac{\partial \theta_i(x_i,t)}{\partial x_i} \right) - N \cdot \frac{\partial^2 y_i(x_i,t)}{\partial x_i^2} - \frac{\bar{m} \cdot I_i}{A_i} \cdot \frac{\partial^2 y_i(x_i,t)}{\partial t^2} = 0 \tag{1.b}$$

$$(0 \le x_i \le L_i) \qquad (i = 1, 2, \ldots, k+1)$$

Where, $y_i(x_i,t)$ represents transverse deflection of the ith beam segment; $\theta_i(x_i,t)$ is the rotation angle due to bending moment of the i[th] beam segment; $\overline{m}_i$ is mass per unit length of the i[th] beam segment; N is the axial compressive force; A_i is the cross-section area of the ith beam segment; I_i is moment of inertia of the i[th] beam segment; L_i is the length of the i[th] beam segment;

$\bar{k}$ is the shape factor due to cross-section geometry of the beam; E, G are Young's modulus and shear modulus of the beam, respectively; x_i is the position of the i[th] beam segment; t is time variable.

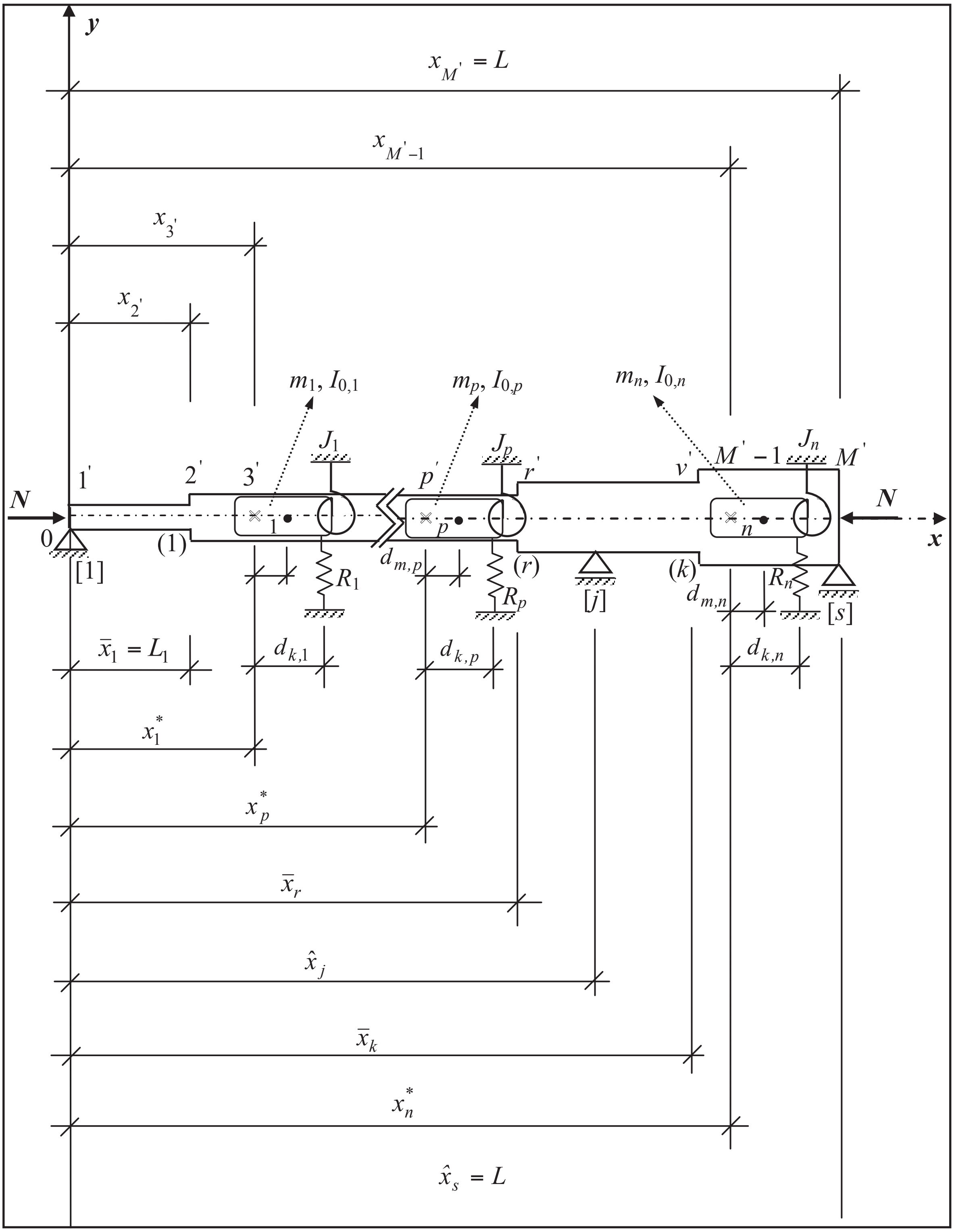


Figure 1. The axial-loaded Timoshenko multiple-step beam with intermediate pinned supports and carrying multiple elastic-supported rigid bars.

The parameters appearing in the foregoing expressions have the following relationships:

$$\frac{\partial y_i(x_i,t)}{\partial x_i} = \theta_i(x_i,t) + \gamma_i(x_i,t) \tag{2.a}$$

$$\bar{M}_i(x_i,t) = EI_i \cdot \frac{\partial \theta_i(x_i,t)}{\partial x_i} \tag{2.b}$$

$$T_i(x_i,t) = \frac{GA_i}{\bar{k}} \cdot \gamma_i(x_i,t) = \frac{GA_i}{\bar{k}} \cdot \left(\frac{\partial y_i(x_i,t)}{\partial x_i} - \theta_i(x_i,t) \right) \tag{2.c}$$

Where, $\bar{M}_i(x_i,t)$ and $T_i(x_i,t)$ are the bending moment function and shear force function of the ith beam segment, respectively, and $\gamma_i(x_i,t)$ is the associated shearing deformation of the i[th] beam segment.

After some manipulations by using Eqs. (1) and (2), one obtains the following uncoupled equations of motion for the axial-loaded Timoshenko multiple-step beam as:

$$\left(1 - \frac{N \cdot \bar{k}}{GA_i}\right) \cdot EI_i \cdot \frac{\partial^4 y_i(x_i,t)}{\partial x_i^4} + N \cdot \frac{\partial^2 y_i(x_i,t)}{\partial x_i^2} + \bar{m}_i \cdot \frac{\partial^2 y_i(x_i,t)}{\partial t^2} - \left(1 + \frac{E \cdot \bar{k}}{G} - \frac{N \cdot \bar{k}}{GA_i}\right) \cdot \frac{\partial^4 y_i(x_i,t)}{\partial x_i^2 \cdot \partial t^2} + \frac{\bar{m}_i^2 \cdot I_i \cdot \bar{k}}{A_i^2 \cdot G} \cdot \frac{\partial^4 y_i(x_i,t)}{\partial t^4} = 0 \tag{3.a}$$

$$\left(1 - \frac{N \cdot \bar{k}}{GA_i}\right) \cdot EI_i \cdot \frac{\partial^4 \theta_i(x_i,t)}{\partial x_i^4} + N \cdot \frac{\partial^2 \theta_i(x_i,t)}{\partial x_i^2} + \bar{m}_i \cdot \frac{\partial^2 \theta_i(x_i,t)}{\partial t^2} - \left(1 + \frac{E \cdot \bar{k}}{G} - \frac{N \cdot \bar{k}}{GA_i}\right) \cdot \frac{\partial^4 \theta_i(x_i,t)}{\partial x_i^2 \cdot \partial t^2} + \frac{\bar{m}_i^2 \cdot I_i \cdot \bar{k}}{A_i^2 \cdot G} \cdot \frac{\partial^4 \theta_i(x_i,t)}{\partial t^4} = 0 \tag{3.b}$$

The general solution of Eq.(3) can be obtained by using the method of separation of variables as:

$$y_i(x_i,t) = \phi_i(x_i) \cdot \sin(\omega \cdot t) \tag{4.a}$$

$$\theta_i(x_i,t) = \theta_i(x_i) \cdot \sin(\omega \cdot t) \quad (0 \le z_i \le L_i/L) \quad (i = 1, 2, ..., k+1) \tag{4.b}$$

in which

$$\phi_i(z_i) = C_{i,1}.\cosh(D_{i,1}.z_i) + C_{i,2}.\sinh(D_{i,1}.z_i) + C_{i,3}.\cos(D_{i,2}.z_i) + C_{i,4}.\sin(D_{i,2}.z_i);$$

$$\theta_i(z_i) = K_{i,3} \cdot C_{i,1}.\sinh(D_{i,1}.z_i) + K_{i,3} \cdot C_{i,2}.\cosh(D_{i,1}.z_i) + K_{i,4} \cdot C_{i,3}.\sin(D_{i,2}.z_i) - K_{i,4} \cdot C_{i,4}.\cos(D_{i,2}.z_i);$$

$$D_{i,1} = \sqrt{\frac{1}{2} \cdot \left(-\beta_i + \sqrt{\beta_i^2 + 4 \cdot \alpha_i^4} \right)};$$

$$D_{i,2} = \sqrt{\frac{1}{2} \cdot \left(\beta_i + \sqrt{\beta_i^2 + 4 \cdot \alpha_i^4} \right)};$$

$$\beta_i = \frac{\left[\frac{N_r \cdot \pi^2 \cdot EI_1}{L^2} + \left(1 + \frac{E \cdot \bar{k}}{GA_i} - \frac{N_r \cdot \pi^2 \cdot EI_1 \cdot \bar{k}}{GA_i \cdot L^2}\right) \cdot \frac{\bar{m}_i \cdot I_i}{A_i} \cdot \omega^2 \right]}{\left(1 - \frac{N_r \cdot \pi^2 \cdot EI_1 \cdot \bar{k}}{GA_i \cdot L^2}\right) \cdot \frac{EI_i}{L^2}}$$

$$\alpha_i^4 = \frac{\lambda_i^4 \cdot EI_i - \frac{\bar{m}_i^2 \cdot I_i \cdot \bar{k} \cdot \omega^4 \cdot L^4}{A_i^2 \cdot G}}{\left(1 - \frac{N_r \cdot \pi^2 \cdot EI_1 \cdot \bar{k}}{GA_i \cdot L^2}\right) \cdot EI_i};$$

$N_r = \frac{N \cdot L^2}{\pi^2 \cdot EI_1}$ (nondimensionalized multiplication factor for the axial compressive force) ;

$\lambda_i = \sqrt[4]{\frac{\bar{m}_i.\omega^2.L^4}{EI_i}}$ (frequency factor) ;

$$K_{i,3} = \frac{GA_i \cdot D_{i,1}}{\bar{k} \cdot \left(-EI_i \cdot D_{i,1}^2 - \frac{\bar{m}_i \cdot I_i}{A_i} \cdot \omega^2 + \frac{GA_i}{\bar{k}} \right)};$$

$$K_{i,4} = \frac{-GA_i \cdot D_{i,2}}{\bar{k} \cdot \left(EI_i \cdot D_{i,2}^2 - \frac{\bar{m}_i \cdot I_i}{A_i} \cdot \omega^2 + \frac{GA_i}{\bar{k}} \right)};$$

$z_i = \frac{x_i}{L}$; $C_{i,1}$, ..., $C_{i,4}$ are the constants of integration; L is total length of the beam; ω is the natural circular frequency of the vibrating system.

The bending moment and shear force functions of the i[th] beam segment with respect to z_i are given below:

$$\bar{M}_i(z_i,t) = \frac{EI_i}{L} \cdot \frac{d\theta_i(z_i)}{dz_i} \cdot \sin(\omega \cdot t) \tag{5.a}$$

$$T_i(z_i,t) = \frac{GA_i}{\bar{k}} \cdot \left(\frac{1}{L} \cdot \frac{d\phi_i(z_i)}{dz_i} - \theta_i(z_i) \right) \cdot \sin(\omega \cdot t) \tag{5.b}$$

3. DETERMINATION OF THE NATURAL FREQUENCIES

The position is written due to the values of the displacement, slope, bending moment and shear force functions at the locations of z_i and t for the i[th] segment of Timoshenko beam, as:

$$\{S_i(z_i,t)\}^T = \left\{\phi_i(z_i)\ \ \theta_i(z_i)\ \ \bar{M}_i(z_i)\ \ T_i(z_i)\right\}\cdot\sin(\omega t) \quad (6)$$

Where, $\{S_i(z_i,t)\}$ shows the position vector of the i[th] beam segment.

If the left-end support of the beam is pinned (as shown in Figure 1), the boundary conditions for the left-end support are written as:

$$\phi_{1'}(z=0)=0 \quad (7.a)$$

$$\bar{M}_{1'}(z=0)=0 \quad (7.b)$$

From Eqs. (4.a) and (5.a), the boundary conditions for the left-end pinned support can be written in matrix equation form as:

$$[B_{1'}]\cdot\{C_{1'}\}=\{0\} \quad (8.a)$$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ K_{1,1} & 0 & -K_{1,2} & 0 \end{bmatrix}\begin{array}{c}1\\2\end{array}\cdot\begin{Bmatrix} C_{1',1} \\ C_{1',2} \\ C_{1',3} \\ C_{1',4} \end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad (8.b)$$

Where $K_{1,1}=\dfrac{EI_1\cdot K_{1,3}\cdot D_{1,1}}{L}$;

$$K_{1,2}=-\frac{EI_1\cdot K_{1,4}\cdot D_{1,2}}{L}$$

In the formula of $K_{1,1}$ and $K_{1,2}$, 1 denotes the 1[st] beam segment.

If the left-end support of the beam is clamped, the boundary conditions are written as:

$$\phi_{1'}(z=0)=0 \quad (9.a)$$

$$\theta_{1'}(z=0)=0 \quad (9.b)$$

From Eqs.(4.a) and (4.b), the boundary conditions for the left-end clamped support can be written in matrix equation form as:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & K_{1,3} & 0 & -K_{1,4} \end{bmatrix}\begin{array}{c}1\\2\end{array}\cdot\begin{Bmatrix} C_{1',1} \\ C_{1',2} \\ C_{1',3} \\ C_{1',4} \end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad (10)$$

If the left-end support of the beam is free, the boundary conditions are written as:

$$\bar{M}_{1'}(z=0)=0 \quad (11.a)$$

$$T_{1'}(z=0)=0 \quad (11.b)$$

From Eqs. (5.a) and (5.b), the boundary conditions for the free left-end can be written in matrix equation form as:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$
$$\begin{bmatrix} K_{1,1} & 0 & -K_{1,2} & 0 \\ 0 & K_{1,5} & 0 & -K_{1,6} \end{bmatrix}\begin{array}{c}1\\2\end{array}\cdot\begin{Bmatrix} C_{1',1} \\ C_{1',2} \\ C_{1',3} \\ C_{1',4} \end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \quad (12)$$

Where, $K_{1,5}=\dfrac{GA_1}{\bar{k}}\cdot\left(\dfrac{D_{1,1}}{L}-K_{1,3}\right)$;

$$K_{1,6}=\frac{GA_1}{\bar{k}}\cdot\left(-\frac{D_{1,2}}{L}-K_{1,4}\right)$$

The boundary conditions for the p[th] elastic-supported rigid bar with rotary inertia in the i[th] beam segment are written by using continuity of deformations, slopes and equilibrium of bending moments and shear forces, as (the station numbering corresponding to the p[th] elastic-supported rigid bar is represented by p'):

$$\phi^L_{p'}(z_{p'})=\phi^R_{p'}(z_{p'}) \quad (13.a)$$

$$\theta^L_{p'}(z_{p'})=\theta^R_{p'}(z_{p'}) \quad (13.b)$$

$$\bar{M}^L_{p'}(z_{p'})-\left(I_{0,p}\cdot\omega^2-J_p+m_p\cdot\omega^2\cdot d^2_{m,p}-R_p\cdot d^2_{k,p}\right)\cdot\theta^L_{p'}(z_{p'}) -\left(m_p\cdot\omega^2\cdot d_{m,p}-R_p\cdot d_{k,p}\right)\cdot\phi^L_{p'}(z_{p'})=\bar{M}^R_{p'}(z_{p'}) \quad (13.c)$$

$$T^L_{p'}(z_{p'})-\left(R_p\cdot d_{k,p}-m_p\cdot\omega^2\cdot d_{m,p}\right)\cdot\theta^L_{p'}(z_{p'}) -\left(R_p-m_p\cdot\omega^2\right)\cdot\phi^L_{p'}(z_{p'})=T^R_{p'}(z_{p'}) \quad (13.d)$$

Where, L and R refer to the left side and right side of the p[th] elastic-supported rigid bar, respectively.

In Section Appendix, the boundary conditions for the p[th] elastic-supported rigid bar with rotary inertia in the i[th] beam segment are presented in matrix equation form.

The boundary conditions for the r[th] step change in cross-section are written by using continuity of deformations, slopes and equilibrium of bending moments, as (the station numbering corresponding to the r[th] step change in cross-section is represented

by r'):

$$\phi^L_{r'}\left(z_{r'}\right)=\phi^R_{r'}\left(z_{r'}\right) \tag{14.a}$$

$$\theta^L_{r'}\left(z_{r'}\right)=\theta^R_{r'}\left(z_{r'}\right) \tag{14.b}$$

$$\bar{M}^L_{r'}\left(z_{r'}\right)=\bar{M}^R_{r'}\left(z_{r'}\right) \tag{14.c}$$

$$T^L_{r'}\left(z_{r'}\right)=T^R_{r'}\left(z_{r'}\right) \tag{14.d}$$

In Section Appendix, the boundary conditions for the r[th] intermediate support are presented in matrix equation form.

If the right-end support of the beam is pinned, the boundary conditions for the right-end support are Written as:

$$\phi_{M'}\left(z=1\right)=0 \tag{15.a}$$

$$\bar{M}_{M'}\left(z=1\right)=0 \tag{15.b}$$

From Eqs. (2) and (3), the boundary conditions for the right-end pinned support can be written in matrix equation form as:

$$\left[B_{M'}\right]\cdot\left\{C_{M'}\right\}=\left\{0\right\} \tag{16}$$

Where,

$$\left\{C_{M'}\right\}^T=\left\{\begin{matrix} C_{M',1} & C_{M',2} & C_{M',3} & C_{M',4}\end{matrix}\right\} \tag{17}$$

$$\left[B_{M'}\right]=\begin{array}{c} \begin{matrix} 4M'_i+1 & 4M'_i+2 & 4M'_i+3 & 4M'_i+4 \end{matrix} \\ \begin{bmatrix} \cosh\left(D_{k+1,1}\right) & \sinh\left(D_{k+1,1}\right) & \cos\left(D_{k+1,2}\right) & \sin\left(D_{k+1,2}\right) \\ K_{k+1,1}\cdot\cosh\left(D_{k+1,1}\right) & K_{k+1,1}\cdot\sinh\left(D_{k+1,1}\right) & -K_{k+1,2}\cdot\cos\left(D_{k+1,2}\right) & -K_{k+1,2}\cdot\sin\left(D_{k+1,2}\right) \end{bmatrix} \end{array} \begin{matrix} q-1 \\ q \end{matrix} \tag{18}$$

If the right-end support of the beam is clamped, the boundary conditions for the right-end support are written as:

$$\phi_{M'}\left(z=1\right)=0 \tag{19.a}$$

$$\theta_{M'}\left(z=1\right)=0 \tag{19.b}$$

If the right-end support of the beam is free, the boundary conditions are written as:

$$\bar{M}_{M'}\left(z=1\right)=0 \tag{20.a}$$

$$T_{M'}\left(z=1\right)=0 \tag{20.b}$$

The boundary coefficient matrixes for the right-end support and free right-end are presented in Eqs. (21) and (22), respectively.

$$\left[B_{M'}\right]=\begin{array}{c} \begin{matrix} 4M'_i+1 & 4M'_i+2 & 4M'_i+3 & 4M'_i+4 \end{matrix} \\ \begin{bmatrix} \cosh\left(D_{k+1,1}\right) & \sinh\left(D_{k+1,1}\right) & \cos\left(D_{k+1,2}\right) & \sin\left(D_{k+1,2}\right) \\ K_{k+1,3}\cdot\sinh\left(D_{k+1,1}\right) & K_{k+1,3}\cdot\cosh\left(D_{k+1,1}\right) & K_{k+1,4}\cdot\sin\left(D_{k+1,2}\right) & -K_{k+1,4}\cdot\cos\left(D_{k+1,2}\right) \end{bmatrix} \end{array} \begin{matrix} q-1 \\ q \end{matrix} \tag{21}$$

$$\left[B_{M'}\right]=\begin{array}{c} \begin{matrix} 4M'_i+1 & 4M'_i+2 & 4M'_i+3 & 4M'_i+4 \end{matrix} \\ \begin{bmatrix} K_{k+1,1}\cdot\cosh\left(D_{k+1,1}\right) & K_{k+1,1}\cdot\sinh\left(D_{k+1,1}\right) & -K_{k+1,2}\cdot\cos\left(D_{k+1,2}\right) & -K_{k+1,2}\cdot\sin\left(D_{k+1,2}\right) \\ K_{k+1,5}\cdot\sinh\left(D_{k+1,1}\right) & K_{k+1,5}\cdot\cosh\left(D_{k+1,1}\right) & K_{k+1,6}\cdot\sin\left(D_{k+1,2}\right) & -K_{k+1,6}\cdot\cos\left(D_{k+1,2}\right) \end{bmatrix} \end{array} \begin{matrix} q-1 \\ q \end{matrix} \tag{22}$$

Where, M_i' is the total number of the intermediate stations and is given by:

$$M_i' = M' - 2 \tag{23.a}$$

With,

$$M' = k + n + s \tag{23.b}$$

In Eq. (23.b), M' is the total number of the stations. In Eqs. (18), (21) and (22), q denotes the total number of equations for integration constants given by

$$q = 2 + 4 \cdot \left(M' - 2\right) + 2 \tag{24}$$

From Eq. (24), it can be seen that; the left-end support of the beam has two equations, each intermediate station of the beam has four equations and the right-end support of the beam has two equations.

In this study, the coefficient matrices for left-end support, each elastic-supported rigid bar and right-end support of the axial-loaded Timoshenko multiple-step beam are derived, respectively. In the next step, the numerical assembly technique is used to establish the overall coefficient matrix for the whole vibrating system as is given in Eq. (25). In the last step, for non-trivial solution, equating the last overall coefficient matrix to zero one determines the natural frequencies of the vibrating system as is given in Eq. (26).

$$[B] \cdot \{C\} = \{0\} \tag{25}$$

$$|B| = 0 \tag{26}$$

4. NUMERICAL ANALYSIS AND DISCUSSIONS

In this study, two numerical examples are considered. For the first numerical example, the first four natural frequencies, $\omega\alpha$ (α = 1, …, 4) and for the second numerical example, the first five natural frequencies, $\omega\alpha$ (α = 1, …, 5) are calculated by using a computer program prepared by the author. In this program, the secant method is used in which determinant values are evaluated for a range (ω_α) values. The (ω_α) value causing a sign change between the successive determinant values is a root of frequency equation and means a frequency for the system.

Natural frequencies are found by determining values for which the determinant of the coefficient matrix is equal to zero. There are various methods for calculating the roots of the frequency equation. One common used and simple technique is the secant method in which a linear interpolation is employed. The eigenvalues, the natural frequencies, are determined by a trial and error method based on interpolation and the bisection approach. One such procedure consists of evaluating the determinant for a range of frequency values, ω_α. When there is a change of sign between successive evaluations, there must be a root lying in this interval. The iterative computations are determined when the value of the determinant changed sign due to a change of 10^{-4} in the value of ω_α..

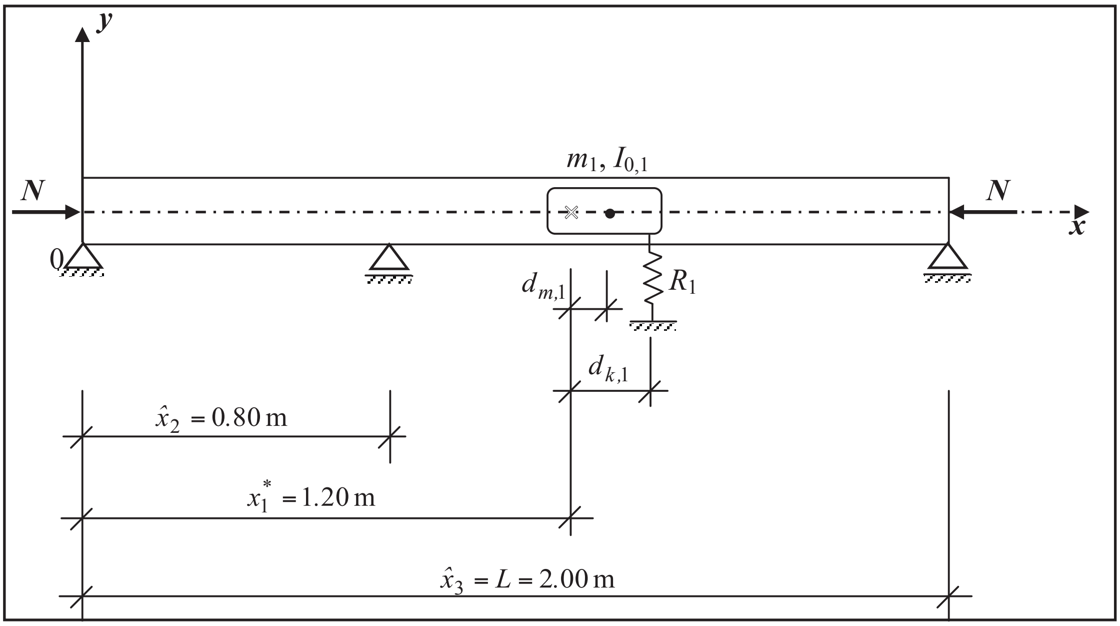
4.1. Free Vibration Analysis of the Axial-Loaded and Two-Span Uniform Timoshenko Beam with an Intermediate Pinned Support and Carrying Single Elastic-Supported Rigid Bar

In the first numerical example (see Figure 2), the pinned-pinned and the clamped-free, the uniform two-span Timoshenko beams with circular cross-section and an intermediate pinned support, and carrying single elastic-supported rigid bar (m_1) with its rotary inertia ($I_{0,1}$) are considered.
In this numerical example, the magnitudes and locations of the elastic-supported rigid bar are taken as: $m_1 = \left(0.80 \cdot \overline{m}_1 \cdot L\right)$ and $I_{0,1} = \left(0.04 \cdot \overline{m}_1 \cdot L^3\right)$ located at $z_1^* = 0.60$, the location of the intermediate pinned support is at $\hat{z}_1 = 0.40$ and those for the linear spring is: $R_1 = 50 \cdot \left(EI_1 / L^3\right)$.

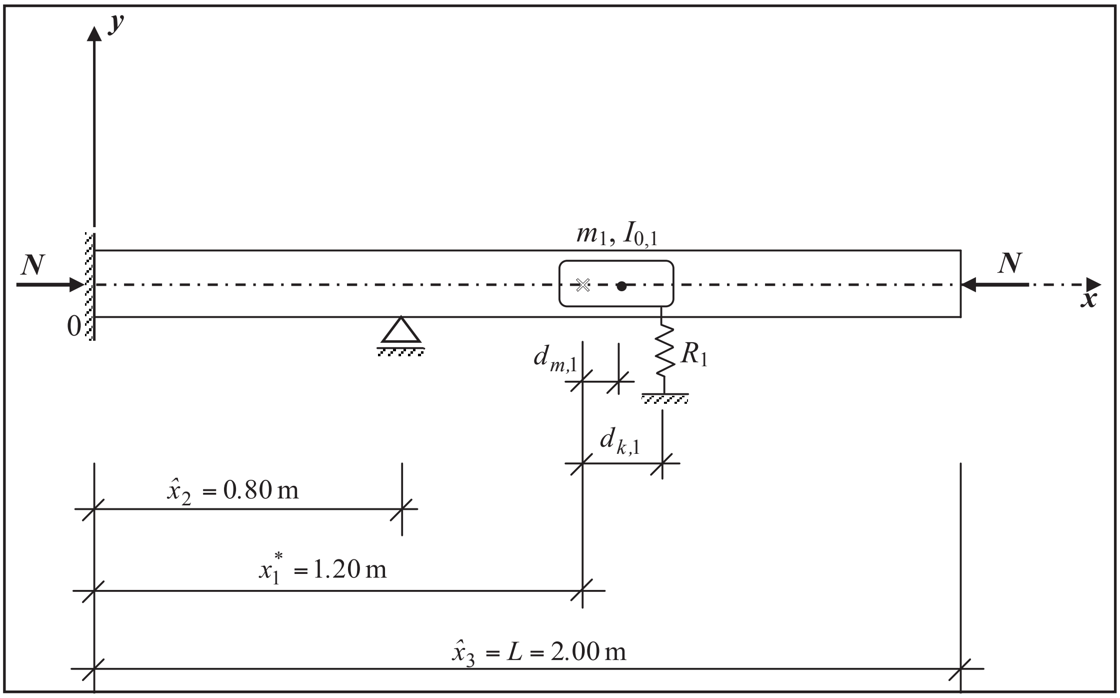
In this numerical example, four different cases are studied. For the first case, $d_{m,1} = d_{k,1} = 0$; for the second case, $d_{m,1} = \left(0.10 \cdot L\right)$ and $d_{k,1} = 0$; for the third case, $d_{m,1} = 0$ and $d_{k,1} = \left(0.15 \cdot L\right)$; for the fourth case, $d_{m,1} = \left(0.10 \cdot L\right)$ and $d_{k,1} = \left(0.15 \cdot L\right)$.

In this numerical example, the mass density of the beam is taken as $\rho = 7.850\times10^3$ kg/m^3; diameter is taken as $d = 0.03$ m; the length of the beam is taken as $L = 2.0$ m; Young's modulus of the beam is taken as $E = 2.069\times10^{11}$ N/m^2; the shear modulus of the beam is taken as $G = 7.95769\times10^{10}$ N/m^2; the shape factor of the beam is taken as $\bar{k} = 4/3$ and the nondimensionalized multiplication factors for the axial compressive force are taken as N_r = 0.0, 0.10 and 0.20, respectively.

The frequency values obtained for the first four modes of the pinned-pinned Timoshenko beam are presented in Table 1, for the first four modes of the clamped-free Timoshenko beam are presented in Table 2 being compared with the frequency values obtained for N_r = 0.0, 0.10 and 0.20.



(a) Pinned-pinned beam.



(b) Clamped-free beam.

Figure 2. The two-span uniform Timoshenko beam with an intermediate pinned support and carrying an elastic-supported rigid bar.

Table 1. The first four natural frequencies of the two-span uniform pinned-pinned Timoshenko beam with an intermediate pinned support and carrying an elastic-supported rigid bar for different values of N_r.

$d_{m,1}$, $d_{k,1}$	ω_α (rad/sec)	(BET) $N_r = 0.00$ (Lin, 2010)	(TBT) $N_r = 0.00$	$N_r = 0.10$	$N_r = 0.20$
$d_{m,1} = 0$, $d_{k,1} = 0$	ω_1	156.1808	156.1181	154.5329	152.9169
	ω_2	308.2504	307.3963	306.1653	304.9265
	ω_3	804.4782	801.0462	797.4270	793.7933
	ω_4	992.0426	988.0663	984.5237	980.9681
$d_{m,1} = 0.10 \cdot L$, $d_{k,1} = 0$	ω_1	129.3295	129.2773	127.9920	126.6821
	ω_2	365.7200	364.7675	363.2020	361.6230
	ω_3	811.9713	808.3999	804.8850	801.3580
	ω_4	983.2062	979.2499	975.6421	972.0204
$d_{m,1} = 0$, $d_{k,1} = 0.15 \cdot L$	ω_1	169.7596	169.7000	168.2116	166.6957
	ω_2	304.7648	303.8993	302.6698	301.4332
	ω_3	804.4183	800.9870	797.3663	793.7312
	ω_4	992.2359	988.2606	984.7201	981.1665
$d_{m,1} = 0.10 \cdot L$, $d_{k,1} = 0.15 \cdot L$	ω_1	140.6334	140.5847	139.3863	138.1665
	ω_2	361.5425	360.5740	359.0008	357.4141
	ω_3	811.8422	808.2720	804.7541	801.2242
	ω_4	983.1896	979.2333	975.6261	972.0050

Table 2. The first four natural frequencies of the two-span uniform clamped-free Timoshenko beam with an intermediate pinned support and carrying an elastic-supported rigid bar for different values of N_r.

$d_{m,1}$, $d_{k,1}$	ω_α (rad/sec)	(BET) $N_r = 0.00$ (Lin, 2010)	(TBT) $N_r = 0.00$	$N_r = 0.10$	$N_r = 0.20$
$d_{m,1} = 0$, $d_{k,1} = 0$	ω_1	59.8369	59.8112	60.3342	60.8634
	ω_2	282.2686	282.0901	281.9945	281.8788
	ω_3	321.4192	320.2889	319.2135	318.1559
	ω_4	1162.5441	1154.8078	1151.9664	1149.1178
$d_{m,1} = 0.10 \cdot L$, $d_{k,1} = 0$	ω_1	53.2546	53.2317	53.6346	54.0412
	ω_2	260.5015	260.3313	260.5849	260.8404
	ω_3	385.0601	383.7341	382.3884	381.0399
	ω_4	1166.9607	1159.1421	1156.3321	1153.5153
$d_{m,1} = 0$, $d_{k,1} = 0.15 \cdot L$	ω_1	77.8948	77.8732	78.4557	79.0446
	ω_2	286.1620	285.9838	285.8016	285.5888
	ω_3	317.8232	316.6821	315.6286	314.6021
	ω_4	1162.5268	1154.7907	1151.9491	1149.1002
$d_{m,1} = 0.10 \cdot L$, $d_{k,1} = 0.15 \cdot L$	ω_1	69.6976	69.6789	70.1167	70.5580
	ω_2	262.7180	262.5495	262.7577	262.9667
	ω_3	380.7431	379.4002	378.0458	376.6890
	ω_4	1166.9235	1159.1050	1156.2946	1153.4773

From Table 1, one can sees that increasing N_r causes a decrease in the first four mode frequency values of the pinned-pinned beam, as expected. From Table 2, increasing N_r causes an increase in the first mode frequency values, a decrease in the other frequency values for the clamped-free beam. The frequency values obtained for the Timoshenko beam without the axial force effect are less than the values obtained for the Bernoulli-Euler beam in the reference (Lin, 2010), as expected, since the shear deformation is considered in Timoshenko Beam Theory.

4.2. Free Vibration Analysis of the Axial-Loaded and Three-Step Timoshenko Beam with An Intermediate Pinned Support And Carrying Three Elastic-Supported Rigid Bars

In the second numerical example (see Figure 3), the pinned-pinned, clamped-pinned and clamped-free Timoshenko beams with circular cross-sections and carrying three elastic-supported rigid bars (m_1, m_2, m_3) with their rotary inertias ($I_{0,1}$, $I_{0,2}$, $I_{0,3}$) are considered. In this example, the magnitudes and locations of the elastic-supported rigid bars and rotary inertias are taken as:

$m_1 = m_2 = m_3 = \left(1.00 \cdot \overline{m}_1 \cdot L\right)$,

$I_{0,1} = I_{0,2} = I_{0,3} = \left(0.01 \cdot \overline{m}_1 \cdot L^3\right)$ located at $z_1^* = 0.10$, $z_2^* = 0.60$ and $z_3^* = 0.85$, the location of the intermediate pinned support is at $\hat{z}_1 = 0.40$ and those for the three linear springs and three rotational springs are:

$$R_1 = R_2 = R_3 = 10 \cdot \left(EI_1/L^3\right)$$
$$J_1 = J_2 = J_3 = 3 \cdot \left(EI_1/L\right)$$

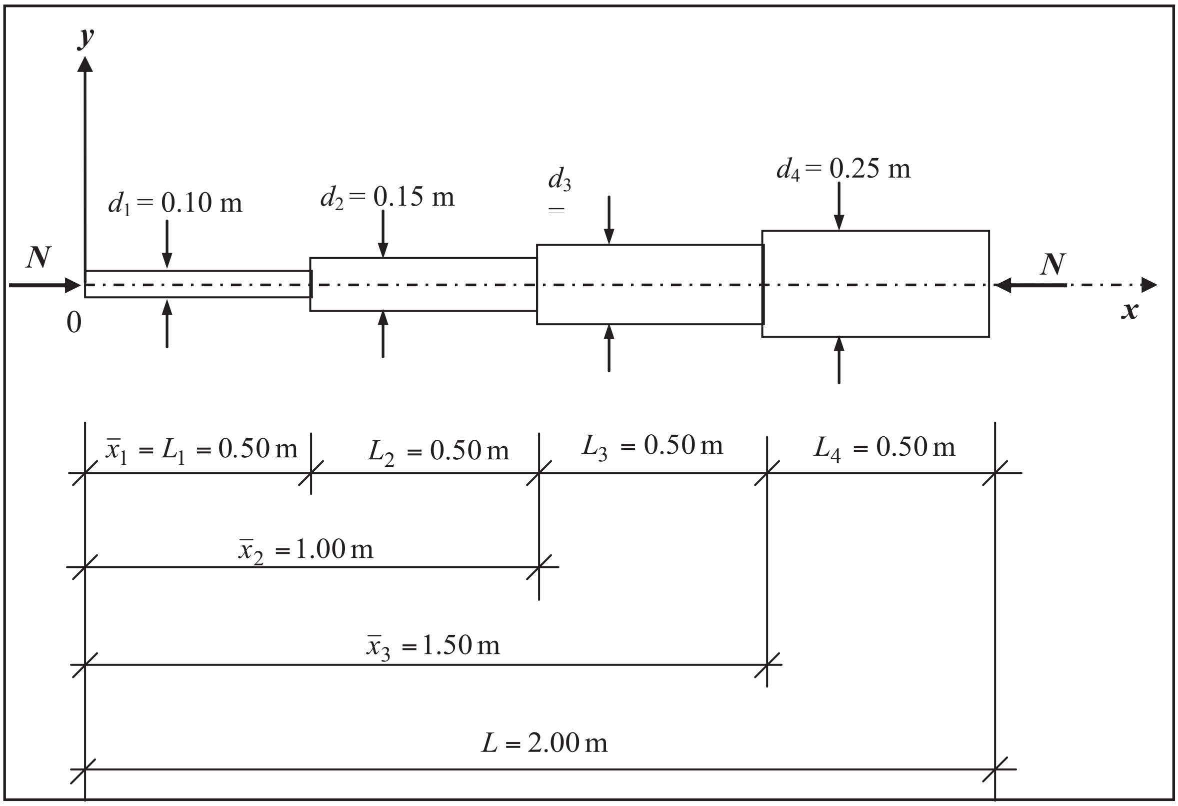


Figure 3. The dimensions of the axial-loaded and three-step Timoshenko beam.

From Figure 3 one sees that, the diameters of the segments are: d_1 = 0.10 m, d_2 = 0.15 m, d_3 = 0.20 m and d_4 = 0.25 m; the lengths of the segments are: $L_1 = L_2 = L_3 = L_4 = 0.50$ m; the locations for the step changes in cross-sections are:

$\bar{z}_1 = 0.25$, $\bar{z}_2 = 0.50$ and $\bar{z}_3 = 0.75$.

In this numerical example, the mass density of the beam is taken as $\rho = 7.8368 \times 10^3$ kg/m³; the length of the beam is taken as $L = 2.0$ m; Young's modulus of the beam is taken as $E = 2.069 \times 10^{11}$ N/m²; the shear modulus of the beam is taken as $G = 7.95769 \times 10^{10}$ N/m²; the shape factor of the beam is taken as $\overline{k} = 4/3$ and the nondimensionalized multiplication factors for the axial compressive force are taken as Nr = 0.0, 0.10 and 0.20.

In this numerical example, four different cases are studied. For the first case, $d_{m,p} = d_{k,p} = 0$; for the second case, $d_{m,p} = (0.08 \cdot L)$ and $d_{k,p} = 0$; for the third case, $d_{m,p} = 0$ and $d_{k,p} = (0.10 \cdot L)$; for the fourth case, $d_{m,p} = (0.08 \cdot L)$ and $d_{k,p} = (0.10 \cdot L)$ (p =1, 2, 3).

The frequency values obtained for the first five modes of the pinned-pinned Timoshenko beam are presented in Table 3, for the first five modes of the clamped-pinned Timoshenko beam are presented in Table 4 and for the first five modes of the clamped-free Timoshenko beam are presented in Table 5 being compared with the frequency values obtained for N_r = 0.0, 0.10 and 0.20.

It can be seen from Tables 3-5 that, as the axial compressive force acting to the beam is increased, the first five natural frequency values of the axial-loaded and three-step Timoshenko beam with pinned-pinned, clamped-pinned and clamped-free boundary conditions are decreased. The frequency values obtained for the Timoshenko beam without the axial force effect are less than the values obtained for the Bernoulli-Euler beam, as expected.

Table 3. The first five natural frequencies of the pinned-pinned Timoshenko beam with three changes in cross-sections and an intermediate pinned support and carrying three elastic-supported rigid bars for different values of N_r.

$d_{m,p}$, $d_{k,p}$ (p = 1, 2, 3)	ω_α (rad/sec)	(BET) N_r = 0.00	(TBT) N_r = 0.00	N_r = 0.10	N_r = 0.20
$d_{m,1} = d_{m,2} = d_{m,3} = 0$	ω_1	1109.5691	1086.7313	1083.5236	1080.2976
	ω_2	1626.4668	1530.0965	1528.3953	1526.6953
	ω_3	2537.7354	2374.2548	2372.6364	2371.0164
$d_{k,1} = d_{k,2} = d_{k,3} = 0$	ω_4	4761.1678	4247.9006	4246.6767	4245.4512
	ω_5	7462.9981	6444.2095	6442.8569	6441.4946
$d_{m,1} = d_{m,2} = d_{m,3} = 0.08 \cdot L$	ω_1	871.6245	853.4848	850.8828	848.2686
	ω_2	1450.9777	1380.6499	1379.6443	1378.6374
	ω_3	3440.0204	3211.2878	3208.4044	3205.5173
$d_{k,1} = d_{k,2} = d_{k,3} = 0$	ω_4	5390.9038	4737.8492	4736.5629	4735.2725
	ω_5	6110.5663	5364.9210	5363.6468	5362.3708
	ω_1	1112.7470	1089.8556	1086.6660	1083.4583
$d_{m,1} = d_{m,2} = d_{m,3} = 0$	ω_2	1628.1663	1531.8123	1530.1017	1528.3923
	ω_3	2537.2420	2373.7865	2372.1705	2370.5529
$d_{k,1} = d_{k,2} = d_{k,3} = 0.10 \cdot L$	ω_4	4760.9513	4247.6049	4246.3811	4245.1557
	ω_5	7463.5397	6444.8882	6443.5343	6442.1708
$d_{m,1} = d_{m,2} = d_{m,3} = 0.08 \cdot L$	ω_1	874.9100	856.8762	854.2824	851.6765
	ω_2	1451.6819	1381.2845	1380.2789	1379.2720
	ω_3	3438.4654	3209.6212	3206.7367	3203.8485
$d_{k,1} = d_{k,2} = d_{k,3} = 0.10 \cdot L$	ω_4	5390.6320	4737.7291	4736.4420	4735.1509
	ω_5	6110.8715	5365.0990	5363.8257	5362.5508

5. CONCLUSIONS

In this study, the frequency values for free vibration of the axial-loaded Timoshenko multiple-step beam carrying multiple elastic-supported rigid beams for different values of axial compressive force. In the two numerical examples, the frequency values are determined for Timoshenko beams with and without the axial force effect and are presented in the tables. The frequency values obtained for the Timoshenko beam without the axial force effect are less than the values obtained for the Bernoulli-Euler beam, as expected, since the shear deformation is considered in Timoshenko beam theory. Generally, the increase in the values of axial force also causes a decrease in the frequency values for all numerical examples.

6. APPENDIX

From Eqs. (4) and (5), the boundary conditions for the pth elastic-supported rigid bar with rotary inertia in the ith beam segment are presented in matrix equation form as:

$$\left[B_{p'}\right] \cdot \left\{C_{p'}\right\} = \{0\} \qquad \text{(A.1)}$$

Where,

$$\left\{C_{p'}\right\}^T = \left\{C_{p'-1,1} \quad C_{p'-1,2} \quad C_{p'-1,3} \quad C_{p'-1,4} \quad C_{p',1} \quad C_{p',2} \quad C_{p',3} \quad C_{p',4}\right\} \qquad \text{(A.2)}$$

Table 4. The first five natural frequencies of the clamped-pinned Timoshenko beam with three changes in cross-sections and an intermediate pinned support and carrying three elastic-supported rigid bars for different values of N_r.

$d_{m,p}$, $d_{k,p}$ (p = 1, 2, 3)	ω_α (rad/sec)	(BET) $N_r = 0.00$	(TBT) $N_r = 0.00$	(TBT) $N_r = 0.10$	(TBT) $N_r = 0.20$
$d_{m,1} = d_{m,2} = d_{m,3} = 0$ $d_{k,1} = d_{k,2} = d_{k,3} = 0$	ω_1	208.4073	204.7075	204.6705	204.6323
	ω_2	1549.7806	1523.0687	1520.8955	1518.7171
	ω_3	2114.2154	1991.7825	1990.7858	1989.7883
	ω_4	3828.2770	3026.2184	3025.1304	3024.0418
	ω_5	5243.2249	4872.9214	4871.6616	4870.3993
$d_{m,1} = d_{m,2} = d_{m,3} = 0.08 \cdot L$ $d_{k,1} = d_{k,2} = d_{k,3} = 0$	ω_1	191.3762	188.1434	188.1173	188.0901
	ω_2	1121.8730	1100.2933	1098.7751	1097.2532
	ω_3	1828.5541	1751.8090	1750.8884	1749.9670
	ω_4	5225.0743	4247.6683	4246.2539	4244.8390
	ω_5	5792.6687	5231.3122	5229.3052	5227.2924
$d_{m,1} = d_{m,2} = d_{m,3} = 0$ $d_{k,1} = d_{k,2} = d_{k,3} = 0.10 \cdot L$	ω_1	214.2382	210.5696	210.5304	210.4901
	ω_2	1553.6548	1526.9453	1524.7783	1522.6060
	ω_3	2115.7440	1993.3019	1992.3051	1991.3075
	ω_4	3827.5292	3025.2652	3024.1769	3023.0880
	ω_5	5243.2952	4872.9677	4871.7081	4870.4462
$d_{m,1} = d_{m,2} = d_{m,3} = 0.08 \cdot L$ $d_{k,1} = d_{k,2} = d_{k,3} = 0.10 \cdot L$	ω_1	196.7482	193.5522	193.5244	193.4955
	ω_2	1124.9707	1103.4637	1101.9491	1100.4309
	ω_3	1829.4408	1752.6323	1751.7123	1750.7915
	ω_4	5224.4015	4246.4075	4244.9927	4243.5773
	ω_5	5792.1167	5231.0885	5229.0816	5227.0689

$$
\left[B_{p'}\right] = \begin{array}{c} \begin{array}{cccccccc} 4p'-3 & 4p'-2 & 4p'-1 & 4p' & 4p'+1 & 4p'+2 & 4p'+3 & 4p'+4 \end{array} \\ \begin{bmatrix} ch_{i,1} & sh_{i,1} & cs_{i,2} & sn_{i,2} & -ch_{i,1} & -sh_{i,1} & -cs_{i,2} & -sn_{i,2} \\ K_{i,3} \cdot sh_{i,1} & K_{i,3} \cdot ch_{i,1} & K_{i,4} \cdot sn_{i,2} & -K_{i,4} \cdot cs_{i,2} & -K_{i,3} \cdot sh_{i,1} & -K_{i,3} \cdot ch_{i,1} & -K_{i,4} \cdot sn_{i,2} & K_{i,4} \cdot cs_{i,2} \\ K_{i,7} & K_{i,8} & K_{i,9} & K_{i,10} & -K_{i,1} \cdot ch_{i,1} & -K_{i,1} \cdot sh_{i,1} & K_{i,2} \cdot cs_{i,2} & K_{i,2} \cdot sn_{i,2} \\ K_{i,11} & K_{i,12} & K_{i,13} & K_{i,14} & -K_{i,5} \cdot sh_{i,1} & -K_{i,5} \cdot ch_{i,1} & -K_{i,6} \cdot sn_{i,2} & K_{i,6} \cdot cs_{i,2} \end{bmatrix} \end{array} \begin{array}{c} \\ 4p'-1 \\ 4p' \\ 4p'+1 \\ 4p'+2 \end{array} \quad \text{(A.3)}
$$

Where,

$$
ch_{i,1} = \cosh\left(D_{i,1} \cdot z_{p'}\right)\ ;\ sh_{i,1} = \sinh\left(D_{i,1} \cdot z_{p'}\right)\ ;\ cs_{i,2} = \cos\left(D_{i,2} \cdot z_{p'}\right)\ ;\ sn_{i,2} = \sin\left(D_{i,2} \cdot z_{p'}\right);
$$

$$
S_{i,1} = I_{0,p} \cdot \omega^2 - J_p + m_p \cdot \omega^2 \cdot d_{m,p}^2 - R_p \cdot d_{k,p}^2\ ;\ S_{i,2} = m_p \cdot \omega^2 \cdot d_{m,p} - R_p \cdot d_{k,p};\ \ S_{i,3} = R_p - m_p \cdot \omega^2;
$$

$$
K_{i,7} = K_{i,1} \cdot ch_{i,1} - K_{i,3} \cdot S_{i,1} \cdot sh_{i,1} - S_{i,2} \cdot ch_{i,1}\ ;\quad K_{i,8} = K_{i,1} \cdot sh_{i,1} - K_{i,3} \cdot S_{i,1} \cdot ch_{i,1} - S_{i,2} \cdot sh_{i,1}\ ;
$$

$$
K_{i,9} = -K_{i,2} \cdot cs_{i,2} - K_{i,4} \cdot S_{i,1} \cdot sn_{i,2} - S_{i,2} \cdot cs_{i,2}\ ;\ K_{i,10} = -K_{i,2} \cdot sn_{i,2} + K_{i,4} \cdot S_{i,1} \cdot cs_{i,2} - S_{i,2} \cdot sn_{i,2}\ ;
$$

$$
K_{i,11} = K_{i,5} \cdot sh_{i,1} - S_{i,3} \cdot ch_{i,1} + K_{i,3} \cdot S_{i,2} \cdot sh_{i,1}\ ;\quad K_{i,12} = K_{i,5} \cdot ch_{i,1} - S_{i,3} \cdot sh_{i,1} + K_{i,3} \cdot S_{i,2} \cdot ch_{i,1}\ ;
$$

$$
K_{i,13} = K_{i,6} \cdot sn_{i,2} - S_{i,3} \cdot cs_{i,2} + K_{i,4} \cdot S_{i,2} \cdot sn_{i,2}\ ;\ K_{i,14} = -K_{i,6} \cdot cs_{i,2} - S_{i,3} \cdot sn_{i,2} - K_{i,4} \cdot S_{i,2} \cdot cs_{i,2}
$$

From Eqs.(4) and (5), the boundary conditions for the r^{th} step change in cross-section are presented in matrix equation form as:

$$
\left[B_{r'}\right] \cdot \left\{C_{r'}\right\} = \{0\} \quad \text{(A.4)}
$$

$$
\left\{C_{r'}\right\}^T = \left\{C_{r'-1,1} \quad C_{r'-1,2} \quad C_{r'-1,3} \quad C_{r'-1,4} \quad C_{r',1} \quad C_{r',2} \quad C_{r',3} \quad C_{r',4}\right\} \quad \text{(A.5)}
$$

$$\left[B_{r'}\right]=\begin{array}{c} \begin{array}{cccccccc} 4r'-3 & 4r'-2 & 4r'-1 & 4r' & 4r'+1 & 4r'+2 & 4r'+3 & 4r'+4 \end{array} \\ \begin{bmatrix} chr_{i,1} & shr_{i,1} & csr_{i,2} & snr_{i,2} & chr_{i+1,1} & shr_{i+1,1} & csr_{i+1,2} & snr_{i+1,2} \\ K_{i,3}\cdot shr_{i,1} & K_{i,3}\cdot chr_{i,1} & K_{i,4}\cdot snr_{i,2} & -K_{i,4}\cdot csr_{i,2} & -K_{i+1,3}\cdot shr_{i+1,1} & -K_{i+1,3}\cdot chr_{i+1,1} & -K_{i+1,4}\cdot snr_{i+1,2} & K_{i+1,4}\cdot csr_{i+1,2} \\ K_{i,1}\cdot chr_{i,1} & K_{i,1}\cdot shr_{i,1} & -K_{i,2}\cdot csr_{i,2} & -K_{i,2}\cdot snr_{i,2} & -K_{i+1,1}\cdot chr_{i+1,1} & -K_{i+1,1}\cdot shr_{i+1,1} & K_{i+1,2}\cdot csr_{i+1,2} & K_{i+1,2}\cdot snr_{i+1,2} \\ K_{i,5}\cdot shr_{i,1} & K_{i,5}\cdot chr_{i,1} & K_{i,6}\cdot snr_{i,2} & -K_{i,6}\cdot csr_{i,2} & -K_{i+1,5}\cdot shr_{i+1,1} & -K_{i+1,5}\cdot chr_{i+1,1} & -K_{i+1,6}\cdot snr_{i+1,2} & K_{i+1,6}\cdot csr_{i+1,2} \end{bmatrix} \end{array} \begin{array}{l} \\ 4r'-1 \\ 4r' \\ 4r'+1 \\ 4r'+2 \end{array} \tag{A.6}$$

Where,

$chr_{i,1} = \cosh\left(D_{i,1}\cdot z_{r'}\right)$; $shr_{i,1} = \sinh\left(D_{i,1}\cdot z_{r'}\right)$; $csr_{i,2} = \cos\left(D_{i,2}\cdot z_{r'}\right)$; $snr_{i,2} = \sin\left(D_{i,2}\cdot z_{r'}\right)$
$chr_{i+1,1} = \cosh\left(D_{i+1,1}\cdot z_{r'}\right)$; $shr_{i+1,1} = \sinh\left(D_{i+1,1}\cdot z_{r'}\right)$; $csr_{i+1,2} = \cos\left(D_{i+1,2}\cdot z_{r'}\right)$;
$snr_{i+1,2} = \sin\left(D_{i+1,2}\cdot z_{r'}\right)$

7. REFERENCES

Chen, D.W. 2003. The exact solutions for the natural frequencies and mode shapes of non-uniform beams carrying multiple various concentrated elements. Structural Engineering and Mechanics. (16), 153-176.

Chen, D.W. and Wu, J.S. 2002. The exact solutions for the natural frequencies and mode shapes of non-uniform beams with multiple spring-mass system. Journal of Sound and Vibration. (255), 299-322.

Gürgöze, M. 1984. A note on the vibrations of restrained beams and rods with point masses. Journal of Sound and Vibration. (96), 461-468.

Gürgöze, M. 1985. On the vibration of restrained beams and rods with heavy masses. Journal of Sound and Vibration. 100, 588-589.

Gürgöze, M. 1996. On the eigenfrequencies of a cantilever beam with attached tip mass and a spring-mass system. Journal of Sound and Vibration. (190), 149-162.

Gürgöze, M. 1998. On the alternative formulation of the frequency equation of a Bernoulli-Euler beam to which several spring-mass systems are attached in-span. Journal of Sound and Vibration. (217), 585-595.

Gürgöze, M. and Batan, H. 1996. On the effect of an attached spring-mass system on the frequency spectrum of a cantilever beam. Journal of Sound and Vibration. (195), 163-168.

Gürgöze, M. and Erol, H. 2001. Determination of the frequency response function of a cantilever beam simply supported in-span. Journal of Sound and Vibration. (247), 372-378.

Gürgöze, M. and Erol, H. 2002. On the frequency response function of a damped cantilever beam simply supported in-span and carrying a tip mass. Journal of Sound and Vibration. (255), 489-500.

Hamdan, M.N. and Jubran, B.A. 1991. Free and forced vibrations of a restrained uniform beam carrying an intermediate lumped mass and a rotary inertia. Journal of Sound and Vibration. (150), 203-216.

Koplow, M.A., Bhattacharyya, A. and Mann, B.P. 2006. Closed form solutions for the dynamic response of Euler-Bernoulli beams with step changes in cross-section. Journal of Sound and Vibration. (295), 214-225.

Lin, H.P. and Chang, S.C. 2005. Free vibration analysis of multi-span beams with intermediate flexible constraints. Journal of Sound and Vibration. (281), 155-169.

Lin, H.Y. 2008. Dynamic analysis of a multi-span uniform beam carrying a number of various concentrated elements. Journal of Sound and Vibration. (309), 262-275.

Lin, H.Y. 2010. An exact solution for free vibrations of a non-uniform beam carrying multiple elastic-supported rigid bars. Structural Engineering and Mechanics. (34), 399-416.

Lin, H.Y. and Tsai, Y.C. 2005. On the natural frequencies and mode shapes of a uniform multi-span beam carrying multiple point masses. Structural Engineering and Mechanics. (21), 351-367.

Lin, H.Y. and Tsai, Y.C. 2006. On the natural frequencies and mode shapes of a multiple-step beam carrying a number of intermediate lumped masses and rotary inertias. Structural Engineering and Mechanics. (22), 701-717.

Lin, H.Y. and Tsai, Y.C. 2007. Free vibration analysis of a uniform multi-span beam carrying multiple spring-mass systems. Journal of Sound and Vibration. (302), 442-456.

Liu, W.H., Wu, J.R. and Huang, C.C. 1998. Free vibration of beams with elastically restrained edges and intermediate concentrated masses. Journal of Sound and Vibration. (122), 193-207.

Naguleswaran, S. 2002a. Natural frequencies, sensitivity and mode shape details of an Euler-Bernoulli beam with one-step change in cross-section and with ends on classical supports. Journal of Sound and Vibration. (252), 751-767.

Naguleswaran, S. 2002b. Vibration of an Euler-Bernoulli beam on elastic end supports and with up to three-step changes in cross-section. International Journal of Mechanical Science. (44), 2541-2555.

Naguleswaran, S. 2002c. Transverse vibrations of an Euler-Bernoulli uniform beam carrying several particles. International Journal of Mechanical Science. (44), 2463-2478.

Naguleswaran, S. 2003a. Transverse vibration of an Euler-Bernoulli uniform beam on up o five resilient supports including ends. Journal of Sound and Vibration. (261), 372-384.

Naguleswaran, S. 2003b. Vibration and stability of an Euler-Bernoulli beam with up to three-step changes in cross-section and in axial force. International Journal of Mechanical Science. (45), 1563-1579.

Naguleswaran, S. 2004a. Transverse vibration and stability of an Euler-Bernoulli beam with step change in cross-section and in axial force. Journal of Sound and Vibration. 270, 1045-1055.

Naguleswaran, S. 2004b. Vibration of an Euler-Bernoulli stepped beam carrying a non-symmetrical rigid body at the step. Journal of Sound and Vibration. (271), 1121-1132.

Wang, J.R., Liu, T.L., Chen, D.W. 2007. Free vibration analysis of a Timoshenko beam carrying multiple spring-mass systems with the effects of shear deformation and rotatory inertia. Structural Engineering and Mechanics. (26), 1-14.

Wu, J.S. and Chen, C.T. 2008. A continuous-mass TMM for free vibration analysis of a non-uniform beam with various boundary conditions and carrying multiple concentrated elements. Journal of Sound and Vibration. (311), 1420-1430.

Wu, J.S. and Chen, D.W. 2001. Free vibration analysis of a Timoshenko beam carrying multiple spring-mass systems by using the numerical assembly technique. International Journal for Numerical Methods in Engineering. (50), 1039-1058.

Wu, J.S. and Chou, H.M. 1999. A new approach for determining the natural frequencies and mode shape of a uniform beam carrying any number of spring masses. Journal of Sound and Vibration. (220), 451-468.

Yesilce, Y. 2010. Effect of axial force on the free vibration of Reddy-Bickford multi-span beam carrying multiple spring-mass systems. Journal of Vibration and Control. (16), 11-32.

Yesilce, Y. and Demirdag, O. 2008. Effect of axial force on free vibration of Timoshenko multi-span beam carrying multiple spring-mass systems. International Journal of Mechanical Science. (50), 995-1003.

Yesilce, Y., Demirdag, O. and Catal, S. 2008. Free vibrations of a multi-span Timoshenko beam carrying multiple spring-mass systems. Sadhana. (33), 385-401.

Zhou, D. 1997. The vibrations of a cantilever beam carrying a heavy tip mass with elastic supports. Journal of Sound and Vibration. (206), 275-279.