A CHARACTERIZATION OF MONTEL SPACES

ABDUL QUDDUS*

SUMMARY: In this paper an analytic criterion for test function spaces of type K { Mp } to be a Montel space are studied.

Key Words: Montel Spaces.

1. INTRODUCTION

In the study of the theory of generalized functions certain classes of function spaces known as test function spaces are discussed. For successful applications it is important that these test function spaces should satisfy, among other properties, the conditions of being Montel spaces. So far various types of test functions have been found to satisfy these conditions. Russian mathematician Gelfand and his colleague Shilov has studied a very general class of test functions. They call these spaces $K\{M_p\}$ spaces of test functions defined on the real space éR with a certain set of weight functions $M_p(x)$. They found a sufficient condition on these weight functions in order to make these spaces into Montel spaces. It is our object to study this condition more closely and try to find necessary and sufficient conditions so that these $K\{M_p\}$ spaces may be a Montel space.

2. DEFINITIONS

The $K\{M_p\}$ space Φ is defined by assigning a sequence of functions $M_p(x)$ satisfying the inequalities $1 \leq M_0(x) \leq M_1(x) \leq ...$, taking a finite or simultaneously infinite values and continuous everywhere they are finite.

By definition, the space $K\{M_p\}$ consisting of all infinitely differentiable functions $\phi(x) = \phi(x_1,...x_n)$, for which the products $M_p(x) D^q \phi(x)$, $(|q| \leq p)$ are everywhere continuous and bounded in the whole space.The norms are defined by the formulas

$$||\phi||_p = \sup_x M_p(x) |D^q \phi(x)|, \quad |q| \leq p \quad (p = 0,1,2...,).$$

Gelfand and Shilov (2) proved that the following condition suffices for the $K\{M_p\}$ space Φ of test functions on $\lceil R^n$ to be a Montel space.

3. PROPERTY (P)

(P) : Given p, there exists p'>p such that

$$\lim_{|x| \to \infty} \frac{Mp(x)}{Mp'(x)} = 0$$

The following property is used in connection with the above condition:

*From Department of Mathematics, Dhaka University, Dhaka-1000, Bangladesh.

4. PROPERTY (A)

A K $\{M_p\}$ space Φ is said to have property (A) if every bounded sequence which is *regularly convergent* to zero is convergent to zero in the topology of Φ.

Now we have a

Theorem 1

If a $K\{M_p\}$ space Φ is a Montel space then it has the *property* (A).

Proof. Let Φ be a Montel space and let $\{\phi_\upsilon\}$ be a bounded sequence in Φ which is regularly convergent to zero. Then $\{\phi_\upsilon\}$ is regularly convergent to 0. Since Φ is a metric space, $\{\phi_\upsilon\}$ contains a convergent subsequence $\{\phi_{\upsilon r}$, r = 1,2.., }. Let the limit of $\{\phi_{\upsilon r}\}$ is the topology of Φ be ϕ_0. Hence, since, the topology of regular convergence is Hausdorff, and since by hypothesis $\{\phi_{\upsilon r}\}$ is regularly convergent to zero, we have $\phi_0 = 0$. Hence, the property (A) holds.

Conversely, let the property (A) hold. Let $A \subset K\{M_p\}$ be a closed bounded subset of Φ. We shall show that A is compact. Let $\phi_\upsilon \,\varepsilon\, A$ (υ = 1,2,...) be an arbitrarily bounded sequence. It is sufficient to show that it contains a convergent subsequence. Since $\{\|\phi_\upsilon\|_1\}$ is bounded, the function

$$\frac{\partial \phi_\upsilon}{\partial x_j} \quad (j = 1, 2, \ldots\ldots, n)$$

are uniformly bounded. Hence, by Arzela's theorem a

subsequence ϕ_{21}, ϕ_{22} ...,for which

converges uniformly $|x| \leq 2$. Continuing thus, we get a bounded sequence ϕ_{11}, ϕ_{22}, ϕ_{33},..., which converges uniformly together with all its derivatives to some function ϕ_0 (x) in any bounded domain. Thus ϕ_υ converges to the element ϕ_0 in the topology of the K $\{M_p\}$ space Φ. This completes the proof of the theorem.

Now we tackle the question of finding a weaker condition than (P) which hopefully will be necessary and sufficient for the K $\{M_p\}$ space Φ to be a Montel space.

First we consider the special case of K {Mp} spaces on the real line satisfying the further restriction:

For each P, $M_p(x) \geq M_p(y)$ ($x \geq y \geq 0$ or $x \leq y \leq 0$).

Let us agree to call such a space a restricted K $\{M_p\}$ space.

5. PROPERTY (p_1)

Given h>0, given a positive integer p, given a sequence $x_\upsilon \geq 0$, there exists $p' \geq p$ such that the sequence

$$\frac{M_{p'}(x_\upsilon + h)}{M_p(x_\upsilon)} \quad (\upsilon = 1,2,\ldots) \text{ is unbounded.}$$

It is clear that (P) implies (P_1).

Theorem 2

If Φ is a restricted K $\{M_p\}$ space, then properties (P_1) and (A) are equivalent. Hence Φ is a Montel space if it satisfies property (P_1).

Proof. Assume that (P_1) holds and that (A) does not hold. Then there exists $M_p(x)$ and a sequence $\{\psi_\upsilon\}$ which is bounded and regularly convergent to zero such that for some k and for some δ>0,

$\sup_x \{M_p(x) | D^k \psi_\upsilon(x) : X \varepsilon \mathbf{R}\} > \delta$.

Replace $D^k \psi_\upsilon$ by ϕ_υ. Since $\{\phi_\upsilon\}$ is regularly convergent to 0, we can find a sequence $\{x_\upsilon\}$ tending either to $+\infty$ or to $-\infty$ such that

$M_p(x_\upsilon) \, |\phi_\upsilon(x_\upsilon)| > \delta$.

Without loss of generality suppose that $\phi_\upsilon(x) > 0$ and that $x_\upsilon \to +\infty$.

By boundedness there exists K > 0 such that

$M_p(x)\, \phi_\upsilon(x) < K$ (all x, all υ)

$M_p(x)\, \phi'_\upsilon(x) < K$ (all x, all υ)

Choose $h = \dfrac{\delta}{2K}$ By (P1), there exists p' such that

$$\frac{M_{p}(x_\upsilon + h)}{M_p(x_\upsilon)} \text{ is unbounded.}$$

If we can prove that $\{M_p(x_\upsilon)\, \phi_\upsilon(x_\upsilon + h)\}$ is bounded below away from zero it will follow that $\{M_{p'}(x_\upsilon + h)\, \phi_\upsilon(x_\upsilon + h)\}$ is an unbounded sequence in contradiction to hypothesis of boundedness in Φ of $\{\phi_\upsilon\}$.

Now $M_p(x)\,\phi_\upsilon(x_\upsilon + h) > M_p(x_\upsilon)\,\phi_\upsilon(x_\upsilon)$

$- M_p(x_\upsilon)\,|\,\phi_\upsilon(x_\upsilon + h) - \phi_\upsilon(x_\upsilon)\,|$

$> \delta - M_p(x_\upsilon)\,|\,\phi'_\upsilon(\xi_\upsilon)\,|\,h\ (x_\upsilon \le \xi_\upsilon \le x_\upsilon + h$

$> \delta - M_p(\xi_\upsilon)\,|\,\phi'_\upsilon(\xi_\upsilon)\,|\,.\,h$

(because $M_p(x)$ is monotonic increasing)

$> \delta - \frac{\delta}{2} = \frac{\delta}{2}$

This completes the proof that property (P_1) implies property (A).

Conversely, assume that (P_1) does not hold. Then there exists h > 0, there exists a positive integer P and without loss of generality

$$\frac{M_{p'}(x_\upsilon+h)}{M_p(x_\upsilon)} \text{ is bounded, say, } \frac{M_{p'}(x_\upsilon+h)}{M_p(x_\upsilon)} \le K_{p'} \qquad (1)$$

$$\text{Let } \phi(x) = \exp\left\{\frac{1}{x^2 - h^2}\right\} \qquad (|x| \le h)$$
$$= 0 \qquad (|x| > h).$$

Let $\phi_\upsilon(x) = a_\upsilon\,\phi(x - x_\upsilon)$ where a_υ is a constant yet to be determined. Then for each k≥0, and for each q≥0, the function $M_q(x)\,|D^k\phi_\upsilon(x)|$ has support inside $[x_\upsilon - h, x_\upsilon + h]$ and attains its maximum at a point $Y_\upsilon(k, q)$ inside $[x_\upsilon, x_\upsilon + h]$. This is because each $M_q(x)$ increases with x and each $|D^k\phi_\upsilon(x)|$ is symmetric about x_υ.

For each k≥0 and each υ

max $|D^k\phi_\upsilon(x)| = a_\upsilon$, *AK*, where

Ak > 0 is independent of υ.

It is clear that for all q, all υ, all k,

$a_\upsilon A_k M_q(x_\upsilon) \le \sup_x\{M_q(x)\,|D^k\phi_\upsilon(x)|\} \le a_\upsilon A_k M_q(x_\upsilon+h)$ (2)

For each υ, choose $a_\upsilon > 0$ so that

$a_\upsilon M_p(x_\upsilon) = 1$ (3)

Then in view of (2), the sequence $\{\phi_\upsilon\}$ fails to converge in Φ.

In fact,

$||\phi_\upsilon||p = \sup\{M_p(x)\,|D^k\phi_\upsilon(x)| : X \varepsilon \mathbb{R},\ 0 \le k \le p\}$

$\ge a_\upsilon A_o M_p(x_\upsilon) = A_o$.

We have now that $\{\phi_\upsilon\}$ is bounded in Φ. In fact, for each k≥0 and each p'≥p,

$\sup_x\{M_{p'}(x)\,|D^k\phi_\upsilon(x)|\} \le a_\upsilon A_k M_{p'}(x_\upsilon + h)$

$\le a_\upsilon A_k M_p(x_\upsilon).K_{p'} = A_k.K_{p'}$

using (1), (2) and (3).

Thus $\{\phi_\upsilon\}$ is bounded and regularly convergent to zero and shows that property (P_1) implies property (A). This completes the proof of the theorem.

Now the restriction that each $M_p(x)$ should be monotonic increasing as $x \to \infty$. It suffices that for each p, there exists x_o such that $M_p(x)$ should be increasing for $x \ge x_o$. If the restriction of monotonicity is removed, the theorem breaks down. For sufficiently pathological $K\{M_p\}$ spaces, property (P_1) can hold and (A) can fail.

The pathology would seem to be characterized by the presence of an infinite number of troughs of $M_p(x)$ becoming increasingly thinner and deeper.

The following counter-example typifies the situation.

6. COUNTER EXAMPLE 3.

The $K\{M_p\}$ space Φ is such that all the $M_p(x)$ are identical with the continuous function M(x) defined as follows:

(i) Mx = M(-x)

(ii)Exception on the closed interval $\left[n, n+\frac{3}{n}\right]$, (n=3,4...)

$$M(x) = e^{x^2}\ (x \ge 0)$$

Now given any x, $M(x+h) \ge \exp\left[(x+\frac{h}{2})^2 - \sqrt{(x+h)}\right]$,

$$M(x) \le e^{x^2}$$

so that $\frac{M(x+h)}{M(x)} \to +\infty$, as $x \to +\infty$.

Let $\phi(x)$ be a suitable C^∞ - function on [0,1] such that, $\phi(0) = 1$, $\phi(1) = 0$, $0 \le \phi(x) \le 1$ $(0 \le x \le 1)$.

For each $\upsilon = 1,2, \ldots$, let $\psi_\upsilon(x)$ be defined as follows:

$\psi_\upsilon(x) = 0$ $(|x| \le \upsilon - 1)$.

$= \phi(\upsilon - |x|)$, $(\upsilon - 1 < |x| < \upsilon)$

$= 1$, $(\upsilon \le |x| \le \upsilon + \frac{1}{\upsilon})$,

$$= \phi[(|x| - (\upsilon + \frac{1}{\upsilon}))], \quad (\upsilon + \frac{1}{\upsilon} < |x| < \upsilon + \frac{2}{\upsilon})$$

$$= 0, \quad (|x| \geq + \frac{2}{\upsilon}).$$

Let $\phi_\upsilon(x) = e^{-\upsilon^2} \psi\upsilon(x)$.

We will show that $\{\phi_\upsilon\}$ is a bounded sequence regularly convergent to 0 which does not converge to 0 in Φ.

Let $a_k = \sup\{|D^k \phi(x)| : 0 \leq x \leq 1\}$.

Clearly $\{\phi_\upsilon\}$ is regularly convergent to 0. Now we show that $\{\phi_\upsilon\}$ is bounded in Φ.

Consider $||\phi_\upsilon||_k = \sup\{M(x)|D^k \phi_\upsilon(x)| : x \varepsilon \mathbf{R}\}$.

We need only consider the range

$$\upsilon - 1 \leq x \leq \upsilon + \frac{2}{\upsilon}$$

For k=0 both $\psi_\upsilon(x)$ and M(x) achieve the maximum in this range at $x=\upsilon$. Hence

$$||\phi_\upsilon|| = M(\upsilon)\, \phi_\upsilon(\upsilon) = a_0 = 1.$$

As a by product we have proved that $\{\phi_\upsilon\}$ does not tend to 0 in Φ. We now estimate $||\phi_\upsilon||_k$ $(k \geq 1)$.

In the range.

$$\upsilon - 1 \leq x \leq \upsilon, \; M(x)\, |D^k \phi_\upsilon(x)| \leq M(\upsilon)\, e^{-\upsilon^2} a_k = a_k$$

In the range, $\upsilon + \frac{1}{\upsilon} \leq x \leq \upsilon + \frac{2}{\upsilon}$,

$$M(x)\left|D^k \phi_\upsilon(x)\right| = e^{\upsilon^2 - \sqrt{\upsilon}} \cdot e^{-\upsilon^2} \cdot \upsilon^k \cdot a_k = a_k \cdot e^{-\sqrt{\upsilon}} \cdot \upsilon^k$$

which tends to 0 as υ tends to infinity.

Thus $||\phi_\upsilon||_k$ is bounded. Hence $\{\phi_\upsilon\}$ is bounded in Φ.

Thus according to the above counter example, *property (P_1)* is not sufficient for $K\{M_p\}$ to be a Montel space in the absence of the monotonivity condition. We shall establish below that property (P_1) is not a necessary condition.

7. In fact the slightly weaker property (P_2), which we now define, fails to be necessary.

Property (P_2).

Given $\varepsilon > 0$ and given P, there exists positive $h < \varepsilon$ such that for any sequence $\{x\upsilon\}$, $x\upsilon \to \infty$ there exists $p' \geq p$ such that

$$\frac{M_{p'}(x_\upsilon + h)}{M_p(x_\upsilon)} \quad \text{is unbounded as } \upsilon \to \infty,$$

Now if Φ is a restricted $K\{M_p\}$ space then (P_1), (P_2) and (P_4) are seen below to be equivalent. In general, (P_1) implies (P_2), although it is not easy to construct examples demonstrating their non-equivalence. The following condition (P_4) is weaker still, and turns out to be necessary in order that Φ be a Montel space.

8. PROPERTY (P_4)

For all h > 0, for any p and for any sequence $\{x_\upsilon\}$, $x_\upsilon \to \infty$ there exists $p' \geq p$ such that

$$\sup\left\{\frac{M_{p'}(x)}{M_p(y)} : x_\upsilon \leq x, y \leq x_\upsilon + h\right\} \text{is bounded as } \upsilon \to \infty.$$

Theorem 4. If Φ is a $K\{M_p\}$ space satisfiying condition (A), then property (P_4) holds.

Proof. Suppose (P_4) does not hold. Then there exists h>0, P>0 and a sequence $x_\upsilon \to \infty$ such that for any $p' \geq p$, there exists $K_{p'} > 0$ such that

$$\sup\left\{\frac{M_{p'}(x)}{M_p(y)} : x_\upsilon \leq x, y \leq x_\upsilon + 2h\right\} < K_{p'}$$

$$\text{Let } \phi(x) = \left\{\begin{array}{ll} \exp\left(\frac{1}{x^2 - h^2}\right), & (|x| < h) \\ 0, & (|x| \geq h) \end{array}\right\}$$

Let $\phi_\upsilon(x) = C_\upsilon\, \phi(x - x_{\upsilon-h})$ where $C_\upsilon > 0$ is a constant to be determined presently.

Let $A_k = \sup_x |D^k \phi(x)|$.

Then for any $k \geq 0$, any υ and $p' > p$,

$\sup\{M_{p'}(x)|D^k \phi(x)| \leq K_{p'} \cdot A_k \cdot C_\upsilon M_p(x_\upsilon + h)$,

$A_0 C_\upsilon M_p(x_\upsilon + h) \leq \sup_x \{M_p(x) \cdot |\phi_\upsilon(x)|\}$.

Choose C_υ so that $C_\upsilon M_p(x_\upsilon + h) = 1$ for all υ.

Then $\{\phi_\upsilon\}$ is bounded regularly convergent to 0, but failing to converge in Φ to 0. Thus property (A) breaks down.

9. COUNTER EXAMPLE 5

To show that (P_2) is not necessary for $K\{M_p\}$ to be a Montel space. The $K\{M_p\}$ space Φ is such that all the $M_p(x)$ are identical with the continuous function M(x) defined as follows:

(i) M(x) = M(-x).

(ii) Except on the closed interval

$[n - h_n, n + h_n]$ (n=1, 2 ...)

$(h_n \geq 0, h_n \to 0)$,

$M(x) = e^{x^2}$.

It is to verify that the property (P_2) breaks down. In fact, for every prescribed $h<\varepsilon$, we can choose $x_\upsilon + h = \upsilon$, since $h_\upsilon \to 0$, x_υ falls outside $[\upsilon-h_\upsilon, \upsilon+h_\upsilon]$ for sufficiently large υ (say $\upsilon \geq N$ and $M_p(x_\upsilon)e^{x_\upsilon^2}$

On the other hand, for $h\upsilon = \frac{1}{2}e^{-2\upsilon^{-2}}$ the topology determined by Mp (x) is the same as that determined by $Np(x) = e^{x^2}$ (all p).

This will follow from the theorem given below. This implies that the space Φ statisfies property (A) and hence Φ is a Montel space.

Let $\Phi_1 = K\{N_p\}$ since $N_p(x) \geq M_p(x)$ it follows that $\Phi_1 \subseteq \Phi$.

To prove that the two topologies are the same it is sufficient to show that $\Phi_1 = \Phi$ since both are complete metrizable spaces and the closed graph theorem holds.

Theorem 6. Let $\Phi = K\{M_p\}$ and $\Phi_1 = K\{N_p\}$ where $M_p(x) \leq N_p(x)$ for each p. Given N_p, if there exists $M_q(x)$ and there exists m such that $N_p(x).g(x,m,q)$ is bounded as a function of x, where

$$f(x,y,m,q) = \left[\sum_{r=0}^{m} \frac{|y-x|^r}{r!}\right] h(y) + \left[\left|\frac{1}{m!}\int_x^y (u-x)^m h(u)\,du\right|\right]$$

$$g(x,m,q) = \inf\left[f(x,y,m,q): -\infty < y < \infty\right], h(x) = \frac{1}{M_q(x)}$$

then $\Phi_1 = \Phi$.

Now before giving the proof, let us look at the counter example given above. Choose m=0.

Case 1

Let $x \; \varepsilon \; [n - h_n, n + h_n]$ for any n.

Then

$$f(x,y) = h(y) + \left|\int_x^y h(u)\,du\right|,$$

$g(x) \leq h(x)$ (taking y = 0).

and $h(x). N_p(x) = 1$.

Hence $g(x). N_p(x)$ is bounded.

Case II

$n - h_n \leq x \leq n + h_n$.

Choose $y = n + h_n$.

$$f(x,y) = \int_x^{x+h_n} h(u)\,du + h(n+h_n)$$

$\leq 2h_n + e^{-(n+h_n)^2}$, since $h(u) \leq 1$

Hence $g(x)\; 2h_n + e^{-(n+h_n)^2}$.

If we put $h_n = \frac{1}{2} e^{-2n^2}$,

then $g(x) \leq 2e^{-(n+h_n)^2}$, so that $Np(x).g(x) \leq 2$.

Thus the above example satisfies the conditions of the theorem.

Proof of the theorem 6.

Let $\phi \varepsilon \Phi$. To prove that $\phi \varepsilon \Phi_1$, it is enough to show that for each p, $N_p(x).\phi(x)$ is bounded as a function of x.

We will show that for all x,

$$\sup\{M_q(y)\,|D^k\phi(y)| : -\infty < y < \infty,\; 0 \leq k \leq m+1\} \geq \frac{|\phi(x)|}{g(x)} \quad (1)$$

Using Taylor's theorem we can write

$$\phi(x) = \sum_{r=0}^{m} \frac{(x-y)^r}{r!}\phi^{(r)}(y) + \frac{1}{m!}\int_y^x (x-u)^m \phi^{(m+1)}(u)\,du \quad (2)$$

since $\phi(x) \varepsilon\; \Phi$,

$$|\phi^{(r)}(y)| \leq ||\phi||_m . h(y), \text{ all } y \quad (3)$$

where $||\phi||_m$ = L.H.S. of (1).

From (2) and (3) we obtain

$$\phi(x) \leq \inf_{y} \left\{ \sum_{r=0}^{m} \left(\frac{|x-y|^r}{r!} \right) h(y) + \frac{1}{m!} \left| \int_y^x (x-y)^m h(u)\, du \right| . \|\phi\|_m \right\}$$

$$= g(x) \, . \, \|\phi\|_m$$

$$\text{i.e. } \frac{|\phi(x)|}{g(x)} \leq \|\phi\|_m \qquad (4)$$

Hence (1) and (4) are the same.

Hence by the given condition

$N_p (x), ½\phi (x)½ \leq \| \phi \|_m . N_p (x) . g (x)$

which is bounded.

Hence $\phi (x) \varepsilon \Phi$.

This is the proof of the theorem.

REFERENCES

1. Gelfand IM and GE Shilov : Generalized Functions, Properties and Operations. Translator: Eugene Saletan, Academic Press, New York and London, vol 1, XVIII+423 pp, 1977.

2. Gelfand IM and GE Shilov : Generalized Functions, vol 2, Spaces of Fundamental and Generalized Functions, p 261, Translator: D Morris Friedman, Amiel Feinstein, P Christian Peltzer. Academic Press, New York and London, 1977.

3. Gould GG and MRA Sardar : A Property of Gelfand- Shilov Test Function Spaces of Type K { Mp }. J London Math. Soc(2), I, pp 545-552. 1969.

Correspondence:
Abdul Quddus
Department of Mathematics,
Dhaka University, Dhaka-1000,
BANGLADESH.