ENERGY BASED METHOD FOR DETERMINATION OF GEOMETRICALLY NON LINEAR FINITE ELEMENT

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SUMMARY: An energy based method is used for the construction of geometrically non linear finite element. A parabolic finite element is constructed as an example to approach continuous arches, mixed structures and semi-circle continuous arches. These examples show the advantage of geometrically non linear elements in terms of reducing the number of finite elements for structural analysis.

Key Words: Non linear finite element.

INTRODUCTION

The displacement method or commonly called Finite Element method is one of the most widely used tools for the analysis of structures (1). This method consists of transfering all the properties of the structure to some particular points called knots, and solving the linear system of equations:

$$AU=F \quad (1)$$

where A is the *stiffness matrix* of the structure, U is the *degrees of freedom* (of the knots) array, F is the force array.

The matrix A is built up from the element stiffness matrices A_i of each geometrically linear beam constituting the whole structure. A_i are customarily determined for bars by analytical methods frequently based on the integration of the slope deflection equations (2,3).

The approximation of curved beams in the finite element method is made by the Discrete Element Model (4), where the beam is divided into a series of linear elements (Figure 1). The accuracy of this methods depends on the number of elements adopted, which goes inversely with cost considerations (The more of elements, the higher is the cost of treatment).

Numeric tests (Figure A.1.1-Table A.1) show that a reasonable accuracy may be obtained with the classic DEM for at least four elements. The idea developed here is to substitute this method by a new one where the geometrically non linear element is considered as one element, i.e. we present an energy based method to built a geometrically non linear finite element which should generalize the classic linear element.

Some applications are made to control the accuracy of a parabolic element when implemented in a classic finite element program and to point out the advantage in terms of reducing the number of elements in finite element structural analysis.

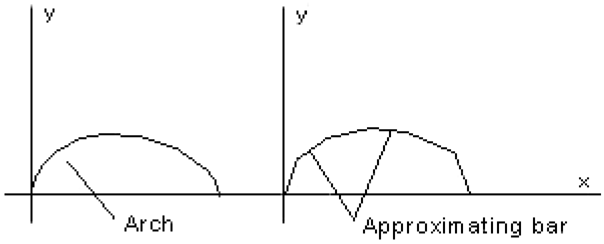

Figure 1: Discrete element model.

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In this paper, an explicit method is proposed for the determinations of the stiffness matrix of an elastic plane arch based on the energy approach (or comp-lementary strain energy) (5). The Arch Model obtained is imple-mented into a finite element code and compared to the Discrete Element Model in terms of performance.

1- Notations

This paragraph presents the condensed notations used in this paper for the formulation of the force method and displacement method. The nuance between strain and complementary strain energy is ignored in those nota-tions.

Let us consider a statically indeterminate arch noted (S), in equilibrium in the OXYZ rectangular coordinates set under the system of load (F).

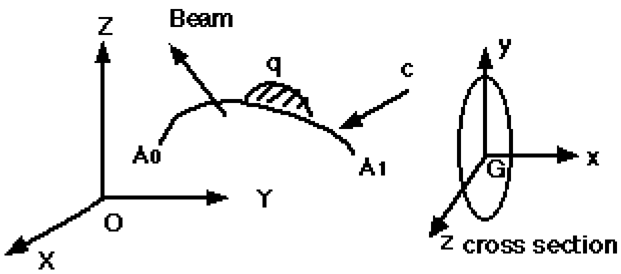


Figure 2

When the arch (S) is released, thus becoming statically determinate (SO), the principle of superposition gives:

$$\sigma = \sigma_f + \bar{\sigma} X \qquad (2)$$

σ_f is the array of the internal forces in SO due to the system (F).

$\bar{\sigma}$ is the (n,n) matrix defined as follows: $\bar{\sigma}_{ij}$ ele-ment is the i^{th} internal force in SO due to j^{th} unit re-dundant force. X is the array of the redundant forces.

1-a-Static approach

The strain energy (6) of the beam may be written:

$$W = \frac{1}{2}\int_0^1 \sigma^t \Lambda^{-1} \sigma ds \qquad (3)$$

Substituting (2) in (3), the energy may be written:

$$W=\frac{1}{2}\int_0^1 X^t\left[\bar{\sigma}^t\Lambda^{-1}\bar{\sigma}\right]X ds+\int_0^1\left[\sigma_f^t\Lambda^{-1}\bar{\sigma}\right]X ds+W_0 \qquad (4)$$

where W_0 is a constant.

The (n,n) matrix defined by:

$$F = \int_0^1\left[\bar{\sigma}^t \Lambda^{-1} \bar{\sigma}\right]ds \qquad (5)$$

is the flexibility (compliance) matrix of the beam.

The (1,n) matrix is $\Delta^t = -\int_0^1\left[\sigma_f^t \Lambda^{-1} \bar{\sigma}\right] ds$ (6)

is a displacement array.

The L.F. Menabrea theorem states that:

$F X = \Delta$ or $X = F^{-1}\Delta$ (7)

1-b Kinematic approach:

Let us call U ((p,1) matrix) the displacement array of the two arch ends.

The relation between Δ and U may be expressed by: $\Delta = KU$ (8)

where K is an (n,p) matrix.

(2) and (8) give: $\sigma = \sigma_1 + \bar{\sigma}F^{-1}KU$ (9)

and then:

$$W=\frac{1}{2}\int_0^1\left[\sigma_f+\bar{\sigma}F^{-1}KU\right]^t\Lambda^{-1}\left[\sigma_f+\bar{\sigma}F^{-1}KU\right]ds \qquad (10)$$

taking into account the expressions of F and Δ, the strain energy may be written:

$$W=-\left[\Delta F^{-1}KU\right]+\frac{1}{2}\left[U^t\left(K^tF^{-1}K\right)U\right]+W_0 \qquad (12)$$

The (p,p) matrix defined by:

$A = K^t F^{-1} K$ (13)

is the *stiffness matrix* and

$f^t = \Delta^t F^{-1} K$ (14)

is the force array.

By substituting (13) and (14) in (12) the expression of the energy becomes:

$$W = \frac{1}{2}U^t A U - f^t U + W_0 \qquad (15)$$

Note that this result is valid for any geometry of the considered beam, namely for arches. The formulation is in effect independent of geometry.

The problem of determining the stiffness matrix of a beam is therefore reduced to the proper determination of the *flexibility matrix F* and the *matrix K.*

We may control the degree of accuracy of the resulting matrix since we may ignore the vanishing terms in the matrix F.

While the compliance matrix is now defined by (5), the matrix K is not. A way to determine this mat-rix is to use its definition and find one by one the arrays Δ; by assuming certain values of Ui. It is clear that this procedure is long.

Remarking that (14) and (7) give:

$f = K^t F^{-1} \Delta = K^t X$ (16)

The matrix K may be determined by writing the equilibrium equations of the beam when it is subjected to the redundant forces X.

2-Determination of the matrices K and F

2-a Determination of F

Let us consider the plane arch defined in the XY plane by the function Y(x)

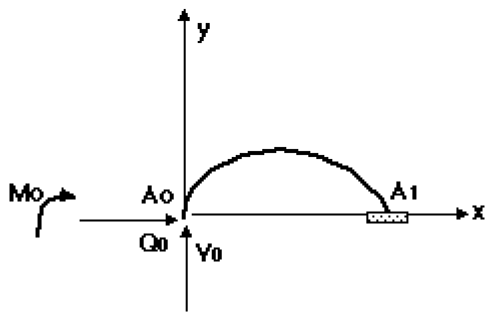


Figure 3

M_0, Q_0 and V_0 being the redundant forces, the internal forces in the arch are:

$N(x) = Q_0 \cos\alpha + V_0 \sin\alpha$

$T(X) = -Q_0 \sin\alpha + V_0 \cos\alpha$ (17)

$M(x) = Q_0 (y-y_0) + V_0 (x-x_0) + M_0$

The matrix σ is given by its definition:

$$\bar{\sigma} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ -(y-y_0) & (x-x_0) & 1 \end{bmatrix}$$

the local operator of elasticity is:

$$\Lambda^{-1} = \begin{bmatrix} \frac{1}{ES} & 0 & \\ 0 & \frac{1}{GSr} & 0 \\ 0 & 0 & \frac{1}{EI} \end{bmatrix}$$

So the compliance matrix is: (18)

$$F = \int_0^1 \begin{bmatrix} \frac{(y-yo)^2}{EI} + \frac{(\cos^2\alpha+\delta\sin^2\alpha)}{ES} & \frac{-(y-yo)(x-xo)}{EI} + \frac{1-\delta}{ES}\sin\alpha\cos\alpha & \frac{-(y-yo)}{EI} \\ \text{sym.} & \frac{(x-xo)^2}{EI} + \frac{(\delta\cos^2\alpha+\delta\sin^2\alpha)}{ES} & \frac{(x-xo)}{EI} \\ \text{sym.} & \text{sym.} & \frac{1}{ES} \end{bmatrix} ds$$

$$\text{where } \delta = \frac{ES}{GSr}$$

when the approximation $\delta = 1$ is made, the above expression becomes: (19)

$$F = \int_0^1 \begin{bmatrix} \frac{(y-y_0)^2}{EI} + \frac{1}{ES} & \frac{-(y-y_0)(x-x_0)}{EI} & \frac{-(y-y_0)}{EL} \\ \text{sym} & \frac{(x-x_0)^2}{EL} + \frac{1}{ES} & \frac{(x-x_0)}{EI} \\ \text{sym} & \text{sym} & \frac{1}{EI} \end{bmatrix} ds$$

The matrix F may be seen as the sum of two matrices:

$F = Fb + Fg$, where (20)

$$F_g = \int_0^1 \begin{bmatrix} \frac{(y-y_0)^2}{EI} & \frac{-(y-y_0)(x-x_0)}{EI} & \frac{-(y-y_0)}{EL} \\ \text{sym} & 0 & 0 \\ \text{sym} & \text{sym} & 0 \end{bmatrix} ds$$

and (21)

$$Fb = \int_0^1 \begin{bmatrix} \frac{1}{ES} & 0 & 0 \\ \text{sym} & \frac{(x-x_0)^2}{EI} + \frac{1}{ES} & \frac{(x-x_0)}{EI} \\ \text{sym} & \text{sym} & \frac{1}{EI} \end{bmatrix} ds$$

Fb is due to the mechanical characteristics of the arch. This matrix is the same as for a bar (Y=0) with the same mechanical properties. Fg is the contribution of the geometric characteristics of the arch. This matrix vanishes when the beam is straight: Fg = 0 if Y = 0.

2-b Determination of the matrix K

When the force array X is applied at the right end of the arch, Figure 4.

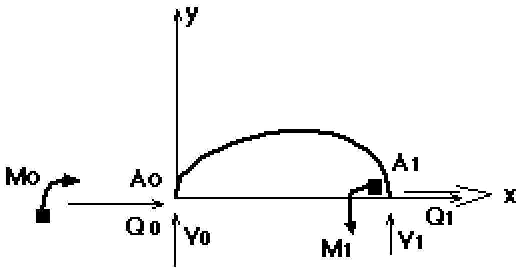


Figure 4

The equilibrium equations are:

M1 - M0 - V0L = 0

Q0 + 01 = 0

v0 + v1 = 0

M1 = M0 + V0 L

Q1 =- Q0 (22)

V1 =-v0

Then the matrix K^t is to be expressed as:

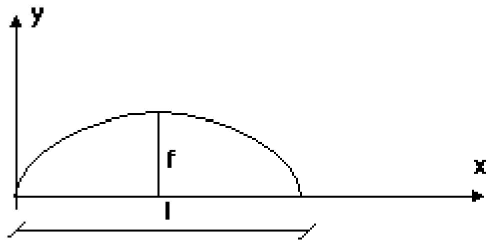
$$K^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & L & 1 \end{bmatrix} \qquad (23)$$

3-Application

3-a-Some comments on the flexibility matrix of a parabolic arch

In this paragraph, the model developed above is implemented in a finite element module (7). Some changes have been introduced to the module to host the new stiffness matrix. The form of the arch adopted is a second degree polynom:

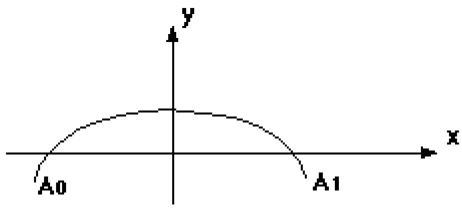
$$y = \frac{4f}{l} x \left(1 - \frac{x}{1}\right) \qquad (24)$$



This curve may be expressed in a particular coordinate set where:

$$\int_{x1}^{x2} y(x)\,dx = 0, \int_{x1}^{x2} x\,dx = 0, \int_{x1}^{x2} xy\,dx = 0 \qquad (25)$$

$$\text{by: } y(x) = -4f\left(\frac{x}{1}\right)^2 + \frac{f}{3} \qquad (26)$$



This transformation of axis is helpful for the integration of the stiffness matrix of symmetric arches.

The flexibility matrix for the above arch is:

$$F = \begin{bmatrix} \frac{8f^2L}{15EI} + \frac{L}{ES} & \frac{-fL^2}{3EI} & \frac{-2fL}{3EI} \\ sym & \frac{L^3}{3EI} + \frac{L}{ES} & \frac{L^2}{2EI} \\ sym & sym & \frac{L}{EI} \end{bmatrix} \qquad (27)$$

Comments: The term $8f^2/15EI$ in the element (1,1) of the flexibility matrix is the geometric flexibility contribution added to the normal flexibility of a bar L / ES.

The geometric flexibility is in general greater than L / ES (Appendix 4).

3-b Case study

The flexibility matrix of the parabolic arch is implemented in a classic finite element FORTRAN program (7) so that it offers two finite element types: the classic bar element and the parabolic element. The user has to declare the parabolic element when needed.

a-Continuous parabolic arch

A continuous parabolic arch (Figure A.2.1) is subjected to a torque at the right end. Table A.2.1 shows that at least 4 elements are needed in the classic method to approach each element of the two span arch, thus 8 elements are needed to approach the two span continuous arch. The use of the parabolic element reduces to 2 the number of elements needed.

b-Continuous semi-circular arch

A continuous semi circular arch (Figure A.3.1) is subjected to a horizontal force at the right end. Table A.3.1 show that while with one element the classic method is unable to describe the displacement at the right end, the parabolic finite element is 80 % accurate, four parabolic elements are enough to approach the structure while eight are needed in the classic method.

c-Mixed structure

In the following example we use the program to analyze a mixed structure composed of linear bars and a parabolic arch:

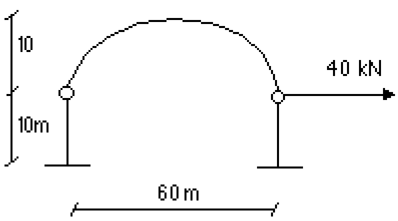


Table A.4.1 shows that only 3 elements are needed in the case of the use of parabolic element, 6 elements are used in the classic method to approach the structure.

In the three span arch frame only 7 elements may be used with the parabolic element and 16 with the classic method.

Table A.4.1

	Theory	Classic 6 elements	Parabolic 3 elements
Moment at right support kNm	36.55	36.33	36.55

4-Conclusion:

-The formalism described in this paper shows the advantage in terms of simplicity of the energy method in the determination of geometrically non linear elements.

-The use of geometrically non linear elements reduce the total number of finite elements and thus the cost of analyses.

-The use of parabolic finite elements reduces by 4 the number of elements of parabolic-arch constituted structures.

A program may include in addition to the classic bar element a library of elements fitting the most common shapes of arches domes etc.

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Appendix.0

Symbols

n: Degree of indeterminacy of the structure, n=3.

p: Number of degrees of freedom of the two ends of the beam: p=6.

s: Length along the curved line.

E: Young modulus.

G: Modulus of elasticity in shear.

S: Area of the cross section.

Sr: Reduced cross section area in shear with respect to the y direction.

I: Moment of inertia with respect to y.

L: Span of the arch.

X: Redundant forces array.

x: x coordinate.

Y(x): The representative function of the arch.

x0,y0: Coordinates of the left end.

α: Angle of the tangent to Y(x) curve.

λx: Normal strain.

λy: Shear strain with respect to y.

ω: Zero line curvature.

σ: The internal forces array:

$$\sigma = \begin{bmatrix} N \\ T \\ M \end{bmatrix}$$

M(s): Bending-moment with respect to y axis.

N(s): Normal force.

T(s): Shear force with respect to y axis.

The HOOKE laws for beams are expressed as:

$$\sigma = \Lambda\varepsilon$$

Λ is the operator of elasticity defined as follows:

$$\Lambda = \begin{bmatrix} ES & & \\ & GS_r & \\ & & EI \end{bmatrix} \quad \text{and} \quad \varepsilon = \begin{bmatrix} \lambda_x \\ \lambda_y \\ \omega \end{bmatrix}$$

is the deformation array.

Note: In the whole text the term 'bar' is used to mean: plane straight line beam.

Appendix.1

Parabolic arch:

Cross section area A=0.08m^2

Moment of inertia I=0.001m^4

Modulus of elasticity E=3 E^{10}Mpa

Force applied at the right end F=40kN.

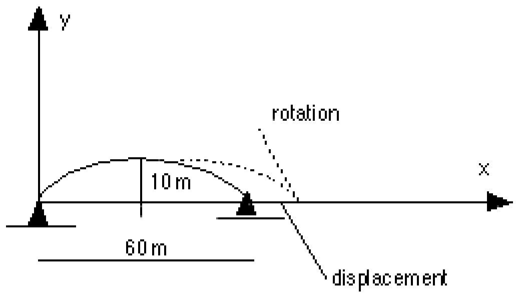


Figure A.1.1

Table A.1

Method	Theory	Arch Model	Discrete Element Model			
Number of elements			1	2	3	4
Displacement Accuracy %	4.26 100	4.26 100	0.001 2.3	2.8 65	3.63 85	3.96 92
Rotation Accuracy %	-0.26 100	-0.26 100	0 0	-0.21 80	-0.24 92	-0.26 99

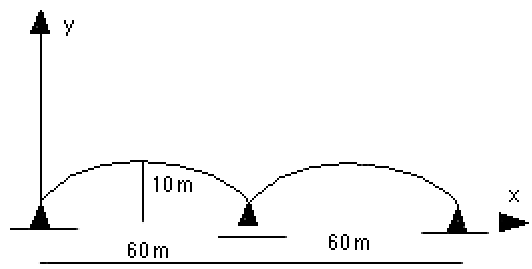
Appendix.2

Continuous arch

Theory

We consider the two span continuous arch defined by the Figure A.2.1 and subjected to a couple at the right end M=100kNm

Figure A.2.1



Span length = 60 m
Cross section area = 0.08 m^2
Moment of Inertia = 0.001 m^4
Modulus of elasticity = 2 104 Mpa.

The degree of indeterminacy of this structure is 2. This structure is analyzed by the force method:

The horizontal reaction at the right end is

$$Q = \frac{5M}{12f} = 4160 \quad N.$$

The bending moment at the central support is

$$m = \frac{M}{6} = 16670 \quad Nm.$$

The central support horizontal displacement is evaluated to 0.388 m.

Table A2.1

Number of elements	Theory	Arch Model	Discrete	Element		Model
		1	1	2	3	4
M 10^4 kNm Accuracy %	1.66 100	1.66 100	-2.5 0	0.71 42	1.25 75	1.42 85
Q 10^4 N Accuracy %	0.416 100	0.416 100	0 0	0.427 97	0.425 98	0.423 98.5
Displacement m Accuracy %	-0.388 100	-0.333 86	0 0	-0.263 67	-0.310 79	-0.326 84

Appendix.3

Semi-circular arch

Theory

We consider the semi-circle arch defined by the Figure A.3.1 and subjected to a horizontal force F=4000N

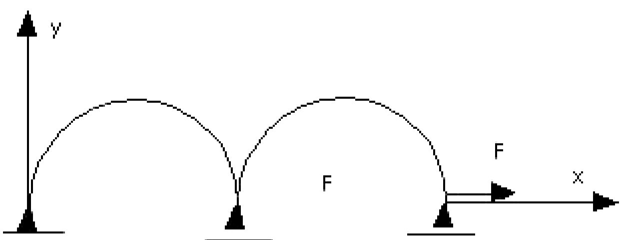


Figure A.3.1

Radius R =100mm

Cross section area =200mm^2.

Moment of Inertia =1660mm^4

Modulus of elasticity =2 10^5 Mpa.

The horizontal displacement of the right support is given by :

$$\Delta = 2 \int_{-R}^{R} Fy^2(x)\,dx = \frac{8FR^3}{3EI} = 32 \ mm$$

Table A3.1: Displacement of the right support compared to theory.

Number of Elements	1	2	3	4	5
Classic (DEM)	0%	71%	93%	96%	91%
Arch Element	80%	95%	89%	88%	91%

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