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Portfolio selection and fractal market hypothesis: Evidence from the London stock exchange

Portföy seçimi ve fraktal piyasalar: Londra borsasından kanıtlar

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Abstract

It is well known that the models supporting the Modern Portfolio Theory (MPT) and the Efficient Market Hypothesis (EMH) are constructed in the framework of random walk theory. However, a large and growing literature criticizes those models. The Fractal Market Hypothesis (FMH) was proposed as an alternative hypothesis to EMH. The motivation of this study is Peters' [45,46] works that examine the portfolio selection case based on the non-normality framework. The aim of the study is to propose a new approach to theoretical framework of portfolio selection in terms of FMH. Daily observations of 92 stocks traded in London Stock Exchange are used to investigate the fractal behavior. Thus, the Hurst exponents as a means of indicator of a fractal structure are calculated for simulated portfolios. Results of the analysis show that the validity of MPT and EMH is questionable in London Stock Exchange. To examine the relationship between Hurst exponents (as a measure of risk) and returns, scattered diagrams are constructed for 5000 simulated portfolios. Existence of a pattern with a frontier is detected that may enable investors to optimize their portfolios. Further, The Hurst exponents of efficient frontier portfolios of Markowitz are calculated in order to investigate whether there is any linkage with the frontier of simulated portfolios. The results show that major deviations occur between these two frontiers. To understand these deviations, the Lyapunov exponents are suggested for detailed information. As a conclusion, it is recommended that investors should calculate an optimal solution with regards to the Hurst and Lyapunov exponents to maximize their returns.

Keywords: Portfolio selection, Efficient frontier, Fractal market hypothesis, The hurst exponent, The lyapunov exponent.

Öz

Modern Portföy Teorisini (MPT) ve Etkin Piyasa Hipotezini (EMH) destekleyen modellerin rastgele yürüyüş teorisi çerçevesinde kurgulandığı bilinmektedir. Ancak, bu modelleri eleştiren geniş ve büyüyen bir literatür, Fraktal Piyasa Hipotezi (FMH) ile EMH'nin geçerliğini sorgulamaktadır. Bu çalışmanın motivasyonu, Peters'ın [45,46] portföy seçimini normal dağılıma uymayan çerçevede inceleyen çalışmalarına dayanmaktadır. Çalışmanın amacı, portföy seçiminin teorik çerçevesine FMH açısından yeni bir yaklaşım önermektir. Çalışmada, fraktal davranışı araştırmak için Londra Menkul Kıymetler Borsası'nda işlem gören 92 hisse senedinin günlük gözlemleri kullanılmıştır. Analizlerde, öncelikle simüle edilmiş portföyler için fraktal yapının bir göstergesi olarak Hurst üsleri hesaplanmıştır. Bulgular, Londra Menkul Kıymetler Borsası'nda MPT ve EMH'nin geçerliliğinin sorgulanabilir olduğunu göstermektedir. Getiriler ve bir risk ölçüsü olarak Hurst üsleri arasındaki ilişkiyi incelemek için 5000 simüle edilmiş portföy oluşturulmuştur. Daha sonra, simüle adilmiş portföyler üzerinde yatırımcıların getirilerini optimize etmelerini sağlayabilecek bir etkin sınırın varlığı tespit edilmiştir. Sonuçları detaylı incelemek amacıyla, simüle edilmiş etkin sınır ile Markowitz'in etkin sınır portföylerinin Hurst üsleri hesaplanmıştır ve karşılaştırmalar yapılmıştır. Sonuçta, bu iki etkin sınır arasında büyük sapmaların meydana geldiğini tespit edilmiştir. Son olarak, sapmaların davranışlarını anlamak için Lyapunov üsleri kullanılmıştır. Araştırma sonucunda, yatırımcıların getirilerini maksimize etmek için Hurst ve Lyapunov üslerine göre optimal bir çözüm hesaplamaları önerilmiştir.

Anahtar kelimeler: Portföy seçimi, Etkin sınır, Fraktal piyasa hipotezi, Hurst üsteli, Lyapunov üsteli.

1 Introduction

Financial analysts' interest in finding the relationship between risk and return goes a long way back to Bachelier [1]. Since then, in finance literature, many efforts have been put on creating models to perceive the behavior of capital flows. Those models are simplifications of reality due to the complex nature of financial markets [2]-[5]. Nevertheless, financial analysts find that their estimations, contrary to their theories, have limited empirical validity [6]-[20]. They realized that a small change in the models have a bigger impact than the theories would predict. It is well known that those models are constructed in the framework of the random walk theory. However, empirical evidence shows that the related data contain outliers. The source of outliers can be assumed to be exogenous variables. But, the existence of outliers may be due to emotions, such as greed and fear, in investment decisions. Moreover, this may result in high volatility which in turn may create a divergence of equilibrium tendency in financial markets. From this point of view, the equilibrium assumption of the Efficient Market Hypothesis (EMH) is redundant.

Before the EMH presented by Fama [4], the Modern Portfolio Theory (MPT) was proposed with specific assumptions involving the Gaussian distribution and the random walk theory [2]. MPT is the concept of diversification in terms of constructing portfolios which minimizes risk for a specified level of returns. The measure of risk is variance of stock returns that are assumed to be a random walk, and independent and identically distributed (IID) variables (For a detailed collected study of random walk characteristic of price behavior can be found in Cootner [21], the random character of stock market prices). In this context, according to the Central Limit Theorem, returns are expected to be normally distributed with finite variance. The MPT was extended by Sharpe [3], Lintner [22],

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and Mossin [23] to Capital Asset Pricing Model (CAPM). The CAPM by combining a riskless asset and the optimal portfolios of the MPT develops a linear measure of the sensitivity of a risky asset to the market risk, called Beta. Thereafter, the CAPM has become a standard of rational investor behavior in financial markets. Later, based on the random walk and IID assumptions, Black and Scholes [24] developed the Option Pricing Model, subsequently, Ross [5] proposed the Arbitrage Pricing Theory (APT). All those models are embraced by the EMH which formulated on the changes in price come only from unexpected new information [69]. The EMH with its three different classifications (weak, semi-strong and strong) evolved from the MPT [4]. Strong form efficiency is considered impossible in the real world [25]. Thus, the weak and semi-strong forms of the EMH are assumed to be applicable in practice [26].

The MPT, CAPM, and EMH have their own successes in financial markets, however, a large and growing literature criticizes the models [26]-[41]. Mandelbrot [42] first challenged the EMH informing those returns are non-normal. Due to non-normality of returns, he stated that the EMH needs to be revised. Essentially, the supporters of the EMH and the MPT ([6],[43],[44], among others) were well aware of the problematic assumptions and the limitations of theories [45]. Consequently, Fractal Market Hypothesis (FMH) was proposed as alternative hypothesis by Peters [45] and Peters [46] to understand the chaotic behavior of financial markets. The FMH emphasizes the impact of liquidity and investment horizons on the behavior of investors. The FMH aims to generate a model for investor behavior and market price movements that fits the real world. A market exists to support a stable or liquid environment for trading. A liquid environment is where the investors with short- and long-horizon come together. Thus, liquidity does not mean trading volume by itself. In this context, liquidity creates stable markets. On the contrary, the EMH does not say anything about liquidity, it says that prices should always be fair whether liquidity exists or not [66]. The EMH assumes there is always enough liquidity. However, markets are not always liquid. When the lack of liquidity strikes, investors are willing to take any price they can, fair or not [46]. This may be considered as the result of panic/courage and fear/greed of investors. These types of situations are the creators of outliers.

The aim of this study is to propose a new approach to theoretical framework of portfolio selection. From this point of view, we suggest several steps consistent with the FMH. As suggested by Peters [45] and Kiehling [47], Hurst exponents are considered as a means of risk measure for portfolios. In the analysis of this study, the Hurst exponent is by itself necessary but not sufficient condition in the portfolio selection problem to avoid the suboptimal portfolios. Therefore, the Lyapunov exponents for the same level of Hurst exponents are considered as an indicator for the best portfolio selection.

The structure of the study is as follows. The second section briefly describes the methodology and the steps of the analysis. The third section conveys some information about the data. The fourth section provides empirical evidence from the analysis. The final section is the conclusion remarks of the study.

2 Methodology

2.1 Theoretical Framework

The EMH's assumptions are mainly summarized as follows: First, investors are rational, therefore, investing activities are uncorrelated. Because the investors are rational, they pay the right price for the (fair) value. Second, changes in price come only from unexpected new information, hence, the distribution of price changes is normal (or Gaussian). Third, transactions are costless, and information is available for every investor [33],[48].

Failure of normality assumption was first realized by Osborne [49]. He plotted the density function of stock market returns, and labeled the returns are "approximately normal". He found out there is more observation in the tails of the distribution then it would be expected. This, fatter tail situation, is the first implication of the departure of the normality assumption. Turner and Weigel [50] studied the volatility of the S&P 500 and Dow Jones index returns, and they found out that daily return distributions are negatively skewed. Moreover, the distributions contain a larger frequency around the mean than the normal distribution should have. Dillén and Stoltz [51] found out that the empirical distribution of stock returns and the residuals are fat tails for twenty stocks quoted on the Stockholm Stock Exchange. Aygoren [17] examined 87 stocks traded in Borsa Istanbul and he concluded that stock price changes do not fit to Normal or Gaussian distribution. Mandelbrot [42] entitled these types of distributions which may have fractal dimensions as "Stable Paretian" that are characterized by undefined, or infinite variance. In Panel A and B of Figure 1, the negatively skewness and fat tails are illustrated, respectively.

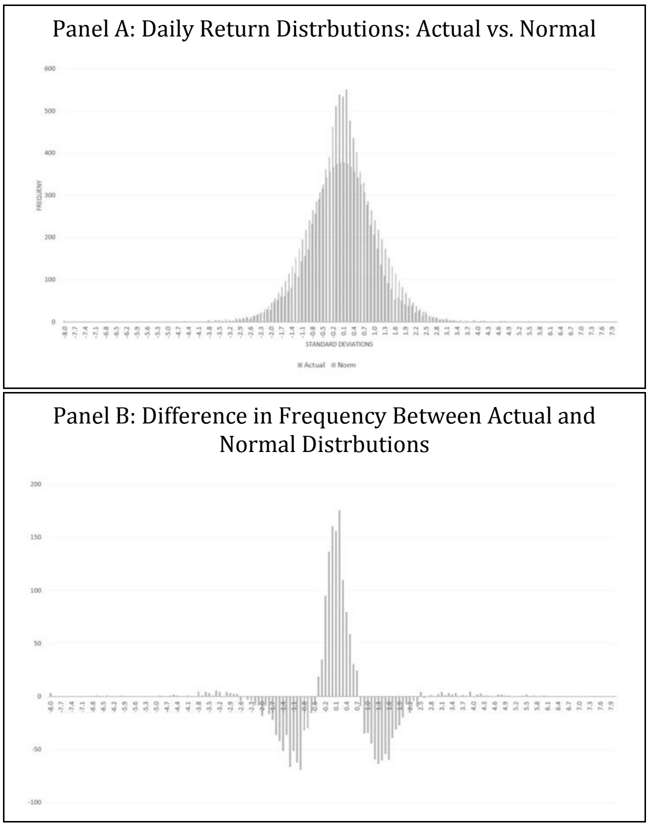

Figure 1. The frequency distribution and difference in frequency of S&P500 stock returns and normal distributions guan [52].

Studies mentioned above present evidence that the stock market returns are not normally distributed. In this regard, the diagnostics of normal distribution (i.e., the correlation coefficient, t-statistics, etc.) is violated, as well as the random walk process in returns is critically weakened.

Based on the debate above, there are several studies which investigate the distributions of returns that fit the real world [26], [53]-[60]. Those studies mainly focus on chaos and fractal behaviors; however, they were confined with individual asset returns. In this study, we aimed to apply chaos and fractal behavior of portfolio returns. We believe that this study will have a theoretical contribution to the finance literature. Even though Peters [46] examined the portfolio selection case based on the non-normality framework, he approached the subject from the point of the single-index model. After several empirical experiences, he indicated that the process should be revisited, and further work should be done.

According to the MPT, portfolio returns are the weighted average of individual expected stock returns. From the point of the fractal behavior, this is the less controversial part of the MPT. But the variance as the measure of risk is an obvious problem because fractal distributions do not have a variance (i.e., undefined, or infinite, variance) to optimize. It is well known that the risk and return tradeoff are a crucial topic for financial investors. In terms of the FMH, the calculation of the expected return of a portfolio with the weighted average of individual expected stock returns is still valid. Yet, to measure the risk new approaches are needed (Tilfani et al. [68] constructed multi-scale portfolios in determining efficient market frontiers using fractal regressions. In their study, covariance matrix is considered as the risk measure in respect to dynamic correlation coefficient [DCC] framework.). In this framework, the fractal dimension may also be evaluated as a risk measure. If a time series has a consistent trend instead of a random walk behavior, it has lower fractal dimensions. The fractal dimension is an interesting alternative for measuring the risk of altering from a real mode and it shows a time path. This feature is different from a measure of dispersion such as the standard deviation [47]. Hurst exponent suggested by Hurst [61] is considered as a means of fractal dimension. In the following section, the theoretical framework of Hurst exponent will be discussed.

2.2 Rescaled range (R/S) analysis and the hurst exponent

Hurst [61] introduced Rescaled Range (R/S) Analysis in the hydrological study of the Nile valley. As a hydrologist Hurst studied on the optimum dam size of the Nile River. He analyzed overflows of the Nile valley for a long period and constructed the R/S Analysis framework.

Calculating the Hurst exponent is a part of R/S Analysis. Let be, a mean of time series $(x_1, x_2, x_3, \dots, x_N)$ is $\bar{y}$.

$$y_t \equiv \ln(x_{k+1}/x_k) \qquad k = 1,2,\dots,N \tag{1}$$

$$\bar{y} = \frac{1}{N}\sum_{t=1}^{j} y_t \qquad j= 1,2,\dots,N \tag{2}$$

On the next step, Y_j, the cumulative time series are calculated.

$$Y_j = \left[(y_1 - \bar{y}) + \dots + (y_j - \bar{y})\right] \tag{3}$$

After calculating the cumulative time series, adjusted range, R_n, can be calculated via the maximum value of Y_j minus the minimum value of Y_j.

$$R_n = \left[\max_{1\le k\le n}\sum_{j=1}^{k}(Y_j - \bar{Y}_n) - \min_{1\le k\le n}\sum_{j=1}^{k}(Y_j - \bar{Y}_n)\right] \tag{4}$$

S_n, the estimated standard deviation with maximum likelihood can be calculated as follows on the next step:

$$S_n = \sqrt{\sum\nolimits_{i=1}^{n}(y_i - \bar{y})^2} \Big/ \sqrt{N} \tag{5}$$

On the last step, the Hurst exponent is adjusted range over standard deviation. Where, c is a constant; H denotes the Hurst exponent; and ${R_n}/{S_n}$ is known as the rescaled range.

$$R_n/S_n = cn^H \tag{6}$$

It is hard to estimate the Eq. (6) because it is an exponential model. A logarithmic conversion is needed:

$$\log(R/S)_n = \log c + H\log n \tag{7}$$

The Hurst exponent, H, may take on values between zero and one. A value of 0.5 is a random walk process. It differs from 0.5 means that a time series' changes are not normally distributed. For a persistent or trend-reinforcing series, it has a value between 0.5 and 1 ($0.5 < H \le 1.0$). The more Hurst exponent approximates 1, the stronger the system's trend-reinforcing behavior gets. Values between 0 and 0.5 ($0 \le H < 0.5$) indicate anti-persistent or mean reverting systems. Moreover, high Hurst values show less noise and clearer trends than lower ones [45],[47],[58],[62],[67]. In Table 1, the fractal taxonomy of times series categorized to understand predictions for meaningful forecasts.

Table 1. The fractal taxonomy of times series.

Term	Color	Hurst Exponent	Fractal Dimension
Anti-persistent, Ergodic, Mean-reverting, Negative Serial Correlation	Pink Noise	$0 < H < 0.50$	$0 \le D < 2$
Gaussian Process, Normal Distribution	White noise	$H \equiv 0.50$	$D \equiv 2$
Brownian Motion, Wiener Process	Brown noise	$H \equiv 0.50$	$D \equiv 2$
Persistent, Trend-reinforcing, Hurst Process	Black noise	$0.50 < H < 1$	$2 > D; D > 1$
Cauchy Process, Cauchy Distribution	Cauchy Noise	$H \equiv 1$	$D \equiv 1$

Source: Mulligan [62].

The Hurst exponent provides information about the persistence of the system. Although the Hurst exponent excepted as a measure of risk, it avoids the information about the length of the prediction horizon or long-memory of the system. There may be many alternatives with the same Hurst exponent but different prediction horizons in investment opportunity set. To choose the best alternative, therefore, the prediction horizon should be calculated. To do so, the Lyapunov Exponent can be a measure of predictability of a system. In the following section, the theoretical framework of the Lyapunov exponent will be discussed.

2.3 The Lyapunov exponent

The Lyapunov exponent characterizes the dynamics of a complex process. It measures the divergence of two neighboring spots after p periods. The Lyapunov exponent is therefore a measure for the predictability of a system. For calculation, an empirical time series $Y = (y_1, y_2, y_3, \dots, y_T)$ m-dimensional phase spaces z could be formed as follow [47]:

$$z_t = (y_t, y_{t+1}, y_{t+2}, \dots, y_{t+m+1})\ t = 1,2, \dots, T - m + 1 \qquad (8)$$

On the next step, all neighboring spots are identified as (a_j, a_k) where $|a_j, a_k| < \varepsilon$ with $a_j \neq a_k$ is true in all conditions. There are N pairs of neighboring spots. The distance between the neighboring spots, δ, in p periods can be calculated as follow:

$$\delta_p^{(j,k)} = \frac{|a_{j+p}, a_{k+p}|}{|a_j - a_k|} \qquad (9)$$

Then the Lyapunov exponent, λ, follows the function:

$$\lambda = \frac{1}{p \times N} \times \sum_{j,k} (ln\delta_p^{(j,k)}) \qquad (10)$$

Negative values of Lyapunov exponents show a contraction in phase space (Phase space is a graph that shows all possible states of a system. In phase space, the value of a variable is plotted against possible values of the other variables at the same time. For instance, if a system has three descriptive variables, the phase space is plotted in three dimensions, with each variable taking one dimension). It means that the distance between two neighboring spots shrinks in the course of time. On the contrary, positive Lyapunov exponents describe a dispersion in phase space [63]. When the Lyapunov exponent grows, the sensitivity of the system reacts rapidly to the change of its starting conditions. From a slightly different perspective, the Lyapunov exponent indicates the loss of predictive ability. The system becomes unpredictable after certain periods of time. Therefore, from the point of financial investors, the Lyapunov exponent can be considered as a measure of prediction horizon length.

Reciprocal of the Lyapunov exponent $(1/\lambda)$ is a way to determine the prediction horizon length (period of a long-memory cycle). In other words, after $1/\lambda$ periods of time, no information about the starting conditions can be found. The less Lyapunov exponent is the longer the prediction horizon length and *vice versa*.

2.4 Steps of the analysis

According to the aim of the study, we construct a methodology using empirical finance, the R/S analysis (Hurst exponent) and the Lyapunov exponent. Analysis of the study involves two sections. *Firstly*, for creating portfolios, uniform distribution weights are generated to simulate the relationship between returns and the Hurst exponents (as a means of risk measure) of portfolios. The steps are as follows (pseudo-codes are available in the Appendix A):

1. Returns of each individual stock are calculated,
2. The weight matrix (involving 5000 weights) is randomly generated from uniform distribution,
3. Using the uniformly distributed portfolio weights and stock returns, the daily *uniform portfolio* return time-series are calculated,
4. Expected returns of each *uniform portfolio* return series are calculated,
5. The Hurst exponents of each of the daily *uniform portfolio* return time-series are estimated by R/S model,
6. The expected returns and the Hurst exponents of *uniform portfolios* are plotted to present the pattern.

Secondly, we are interested in calculating weights of optimal portfolios of Markowitz mean-variance method to examine the relationship between returns and the Hurst exponents of those portfolios. The steps are as follows (pseudo-codes are available in the Appendix B):

Returns of each individual stock are calculated.

1. The weight matrix (involving 5000 weights) is generated by Markowitz mean-variance method. *Frontier portfolios* are the optimal portfolios generated on the efficient frontier,
2. Using the frontier portfolio weights and stock returns, the daily *frontier portfolio* return time-series are calculated,
3. Expected returns of each *frontier portfolio* return series are calculated,
4. The Hurst exponents of each of the daily *frontier portfolio* return time-series are estimated by R/S model,
5. The expected returns and the Hurst exponents of *frontier portfolios* are plotted to present the pattern,
6. The Lyapunov exponents of each daily *frontier portfolio* return time-series are calculated,
7. The expected returns, the Hurst exponents, and the Lyapunov exponents of each daily *frontier portfolio* return time-series are plotted.

3 Data

Daily observations of 92 stocks traded in London Stock Exchange (FTSE-100) are used to investigate behavior of FTSE for the period between January 4, 2010, and November 22, 2019. The stocks that have available data during the study period are selected and the number of observations for each stock is 2580. The dataset is obtained from Bloomberg Professional Database. In this study natural logarithmic price changes are considered as the main data, and they are calculated as follows [64]-[65]:

$$R_t = \ln(P_t) - \ln(P_{t-1}) \qquad (11)$$

Where, P_t is the price of individual stock at the time t; P_{t-1} is the price of individual stock at the time t-1 and R_t is the natural logarithmic price changes or returns of individual stocks. The summary descriptive statistics of dataset are shown in Table 2. It is seen that the behavior of price changes has fat tails and negatively skewed.

Table 2. The summary descriptive statistics of the dataset.

	Minimum value of Observations	Maximum value of Observations
Kurtosis	1.9797	1512.4307
Skewness	-33.7935	0.5207
Minimum of Minimums	-1.8917	-
Maximum of Maximums	-	0.3772
Standard Deviation	0.0109	0.0426
Mean	-0.0005	0.0013

Source: Calculated by Authors.

4 Findings

This study provides an investigation to detect if the empirical evidence from the London Stock Exchange forms a pattern between returns and the Hurst exponents of *uniform portfolios*. Interestingly, we confront a specific pattern between two variables. Figure 2 illustrates that 5000 simulated *uniform portfolios* form a frontier. In this context, the Hurst exponents as a means of risk measures can be used to find out the optimal portfolios. Dotted and solid curves show the whole frontier formed by 5000 simulated *uniform portfolios*. However, the solid curve is of prime importance in terms of optimal *uniform portfolios*, *i.e.,* for a specific level of the Hurst exponent, there is only one optimal solution (the highest return portfolio). To find the optimal portfolio, weight matrix must be calculated. To do so, there needs to be an objective portfolio Hurst exponent function that involves relationship among individual stocks' Hurst exponents. Unfortunately, we do not have the objective function, and it is out of this study's scope. Therefore, this problem can be a topic of further studies.

The above inferences guide us to question if the *frontier portfolios* of Markowitz mean-variance have the same patterns with their Hurst exponents. To achieve this goal, we acquire the weights of 5000 mean-variance frontier portfolios (Figure 3).

Using the frontier portfolio weights and stock returns, the daily *frontier portfolio* return time-series are calculated. Afterwards, the expected returns and Hurst exponents of each *frontier portfolio* return time-series are generated and plotted in Figure 4. Our expectation (the dotted curve in Figure 4) was to detect the same pattern with Figure 2, however, Figure 4 presents a different pattern (solid behavior) far from that. To sum up, there is an obvious difference between portfolio selection of the MPT and fractal structure of financial markets. Figure 4 shows that a sharper decrease in returns occurs as the Hurst exponents increases compared to our expectation. The reason of this behavior may be due to existence of many alternatives with the same Hurst exponent but different prediction horizons in investment opportunity set. To understand these deviations, the Lyapunov exponents of *frontier portfolios* can suggest us a detailed information.

In Section 2.3, we mentioned that reciprocal of the Lyapunov exponent ($1/\lambda$) is a way to determine the predictive ability. Figure 5 illustrates the expected returns, Hurst exponents, and prediction horizon lengths ($1/\lambda$) for *frontier portfolios*. There are positive and negative signs of Lyapunov exponents which also effect the sign of $1/\lambda$. A positive Lyapunov exponent measures "stretching" in phase space; that is, it measures how rapidly nearby neighbor points diverge from one another. On the other hand, a negative Lyapunov exponent measures contraction, how long it takes for a system to reestablish itself after it has been perturbed. Peters [45] states that economic time series contain all the phases of the system, not just the chaotic ones. Therefore, the parameters must be chosen to maximize the measurement of the stretching of points in phase space while minimizing the contractions, that can occur when market activity is truly random or when market activity is low. In this context, the negative Lyapunov exponents of Figure 5 may indicate low market activity periods. This can imply that during the low market activity periods, portfolio selection of the Markowitz's mean-variance approach can mislead the investors. The positive Lyapunov exponents of Figure 5 have another story. They may provide us to select the best investment set in financial markets; that is, the portfolios which have short prediction horizon lengths should have relatively higher returns than long prediction horizon lengths. Thus, investors should create portfolios with optimal prediction horizon lengths for the same level of Hurst exponents. This optimization problem may be another topic for further studies.

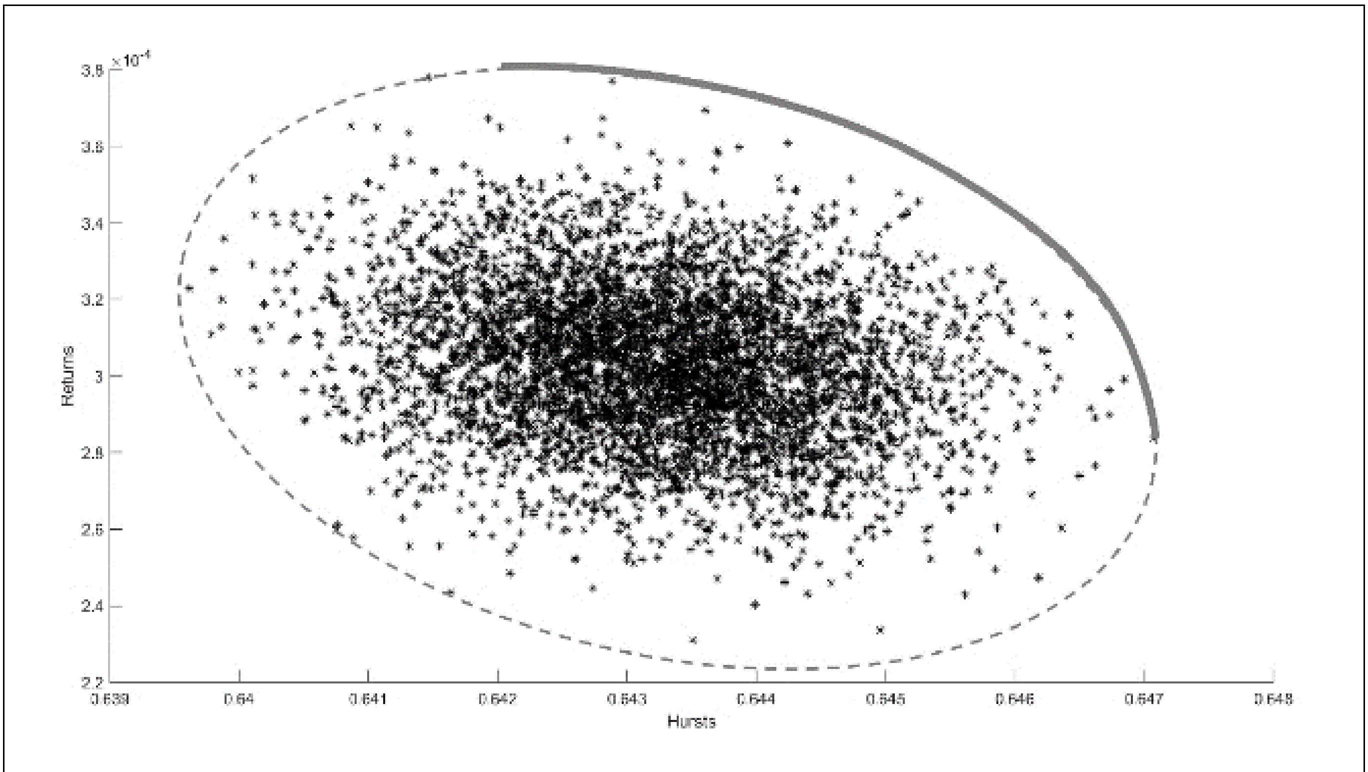

Figure 2. The expected returns and hurst exponents of uniform portfolios.

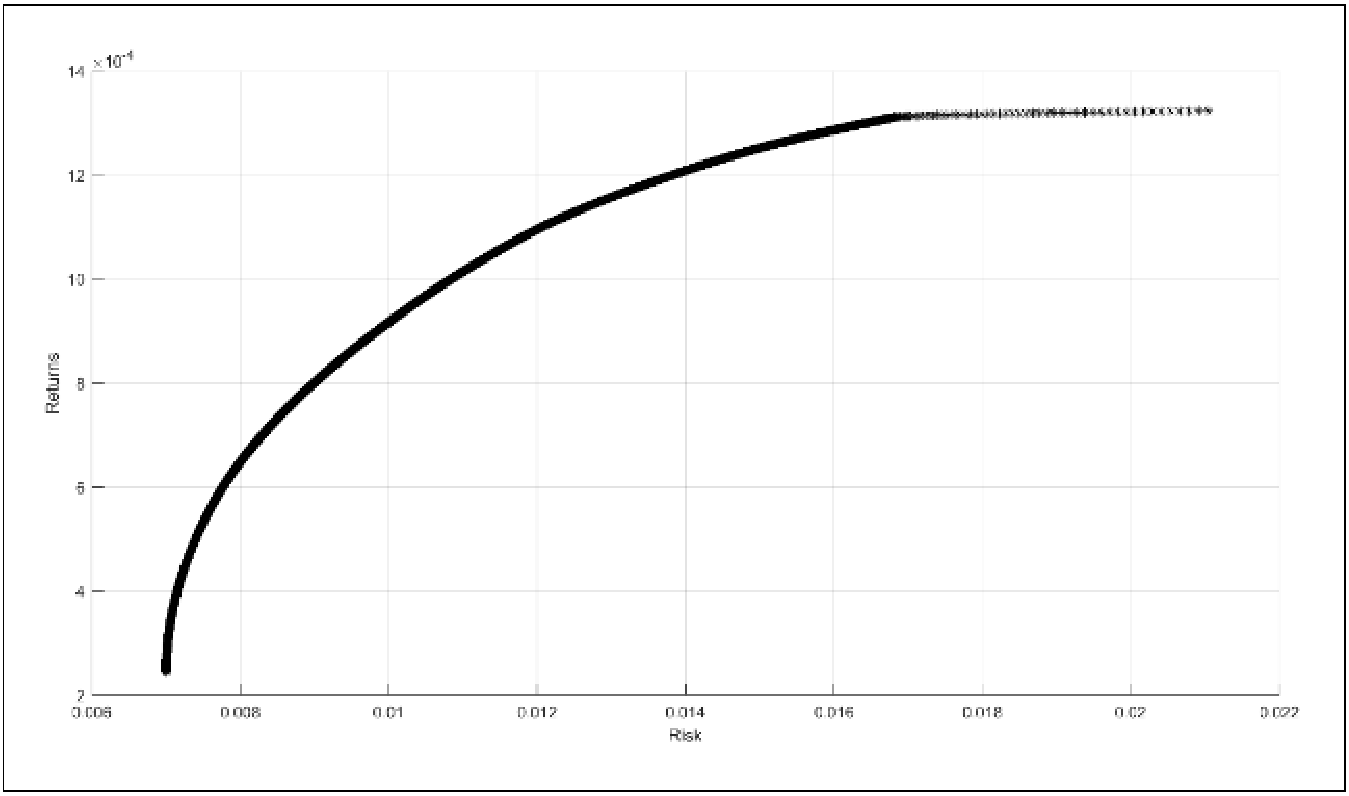

Figure 3. The efficient frontier graph of 5000 frontier portfolios.

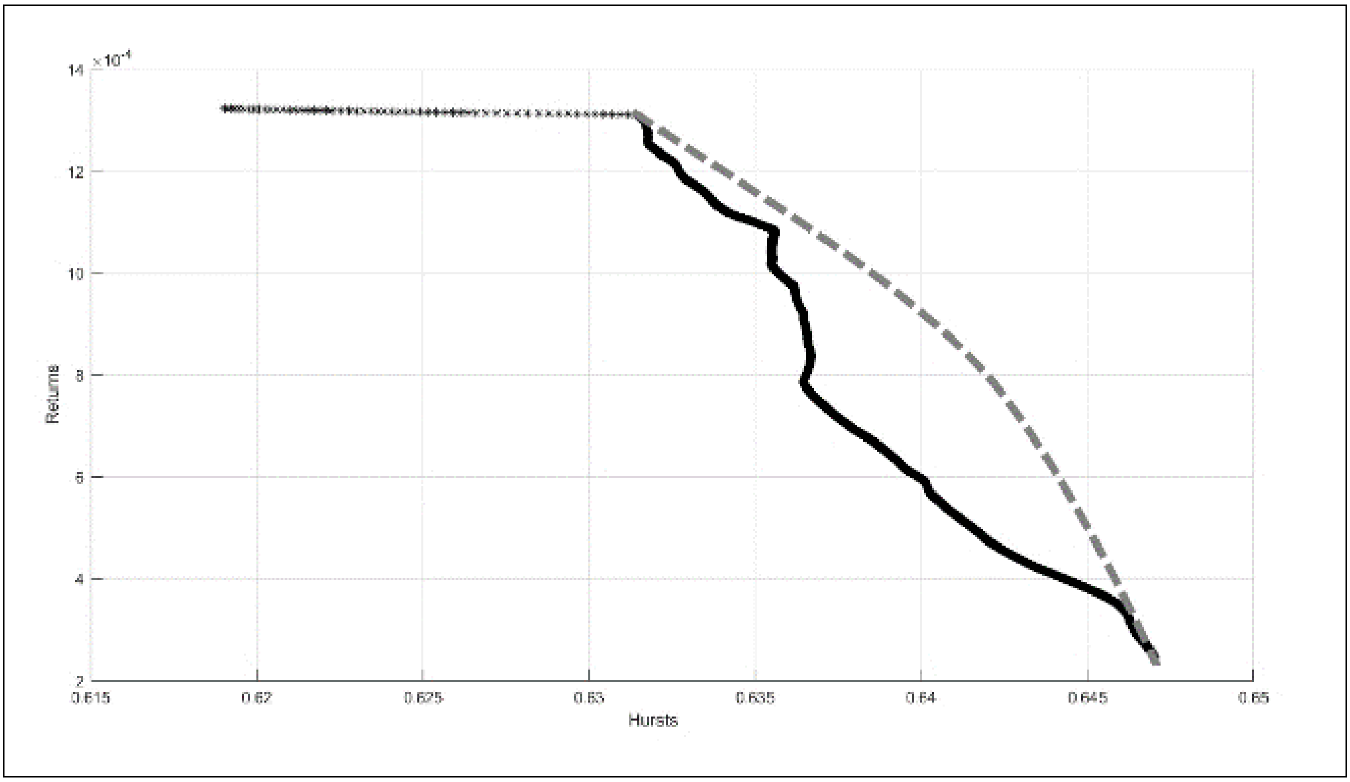

Figure 4. Expected returns and hurst exponents of 5000 frontier portfolios.

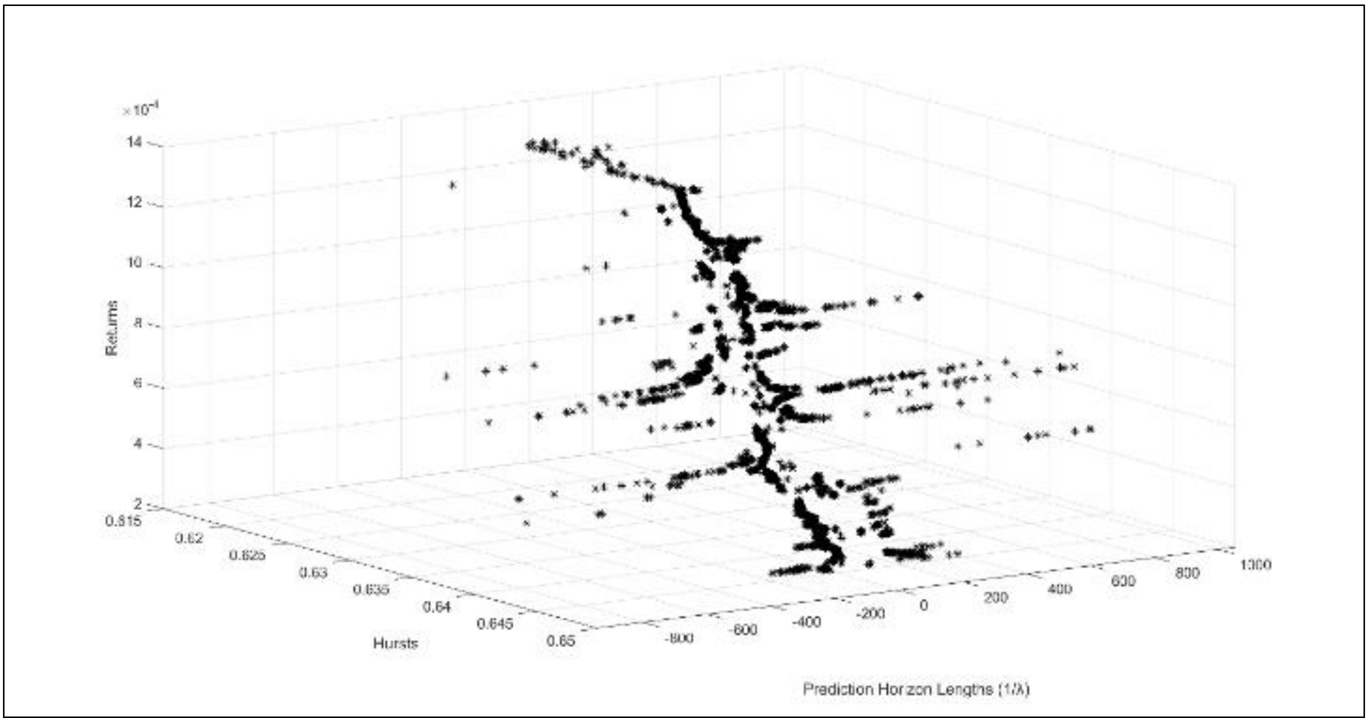

Figure 5. The expected returns, hurst exponents, and lyapunov 1/lambdas of 5000 frontier portfolios.

5 Conclusion

Research of the stock returns' behavior date back to the beginning of the twentieth century and have been an important area in finance literature. Earlier studies define the return behaviors as random walk or Brownian motion. Due to the complex nature of financial markets, those studies can be characterized as the simplifications of reality. Nevertheless, researchers find that their estimations, contrary to the theories, have limited empirical validity, *i.e.,* empirical evidence show that the related data contain outliers. The existence of outliers may be due to the emotions, such as panic/courage and fear/greed of investors, in their investment decisions. According to the Efficient Market Hypothesis (EMH), there are three different classifications of financial markets: weak, semi-strong and strong. Strong form efficiency is considered impossible in the real world due to the existence of outliers. Thus, the weak and semi-strong forms are assumed to be applicable in practice. However, the applicability of weak and semi-strong forms in the real world was intensively criticized by the researchers. Consequently, the Fractal Market Hypothesis (FMH) was proposed as an alternative hypothesis in the sense of the criticism to the EMH. The FMH aims to generate a model for investors' behavior and market price movements that fits the real world.

The motivation of this study is Peters' [45]-[46] works that examine the portfolio selection case based on the non-normality framework. He approached the subject from the point of the single-index model. After several empirical experiences, he indicated that the process should be revisited, and further work should be done. The aim of this study is to propose a new approach to theoretical framework of portfolio selection. From this point of view, we suggest several steps consistent with the FMH. As suggested by Peters [45] and Kiehling [47], Hurst exponents are considered as a means of risk measure for portfolios. In the analysis of this study, we realized that the Hurst exponent is by itself necessary but not sufficient condition in the portfolio selection problem to eliminate the suboptimal portfolios. To avoid the selection of suboptimal portfolios from the investment opportunity set, the Lyapunov exponents for the same level of Hurst exponents are considered as an indicator of the best portfolio.

For empirical analysis, the data involves daily observations of 92 stocks traded in London Stock Exchange (FTSE-100) for the period between January 4, 2010, and November 22, 2019. The stocks that have available data during the study period are selected. The data after December 2019 is not included to the analysis because the effect of coronavirus. For this reason, this period is excluded from the analysis, but we are aware of the coronavirus effect that is supporting to FMH due to creating illiquid markets. Further research should include the data of the period after the coronavirus effect vanishes.

In conclusion, the results of the study have several theoretical contributions. As mentioned, earlier studies focused on individual financial instruments' returns in terms of the FMH. But, in this study, portfolio returns are examined as a new approach to FMH, and we believe that a gap is filled in the finance literature. Findings indicate that there is the existence of the efficient frontier relationship between portfolio returns and the Hurst exponents. It is possible to optimize returns according to the Hurst exponents. However, an objective function is needed for the optimization. The results also show that there is an obvious difference between portfolio selection of the MPT and FMH. A sharper decrease in returns occurs as the Hurst exponents increases compared to the theoretical expectation. To understand these deviations, the Lyapunov exponents are suggested for the detailed information.

Furthermore, the MPT and EMH are invalid in London Stock Exchange. This result is consistent with the literature which criticizes those theories. Secondly, the findings suggest that the portfolio selection of Markowitz's mean-variance approach can mislead the investors. Investors should calculate an optimal solution with regards to the Hurst and Lyapunov exponents. These inferences can be summarized as the empirical implications of this study.

Considering the analysis' results, this study constitutes new research topics in finance literature. Further studies may be to find a mathematical function between the Hurst exponents of individual stock returns and the Hurst exponents of portfolio returns. Another research topic may be to focus on Lyapunov exponents with respect to portfolio selection.

6 Author contribution statements

In the scope of this study, Hakan AYGÖREN and Umut UYAR are equally contributed to the formation of the idea, the design, and the literature review, supplying the materials used and examining the results and the spelling, and checking the article.

7 Ethics committee approval and conflict of interest statement

There is no need to obtain permission from the ethics committee for the article prepared. There is no conflict of interest with any person or institution in the article prepared.

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Appendix A

Algorithm: Generating Uniform Portfolios
P_t : The daily price of individual stock at the time t, $t = 1, \dots, T$
N : Total number of individual stocks
w_i : The portfolio weight of i^{th} stock

Begin
function (Generate returns of individual stocks (R_t)) {
calculate Returns matrix (R_t) using Equation (11),
The size of matrix is $Nx(T-1)$
end
}

Appendix A: Continued.

```
        function (Create weight matrix from uniform distribution) {
            for sum(w_i) = 1
            calculate 5000 uniformly distributed random numbers for each individual stock within the range of [0,1]
            end
            }
            The size of matrix is 5000xN
            end
            }
        function (Calculate uniform portfolio return series) {
            calculate Daily uniform portfolio return matrix by R_t × w_i
            The size of matrix is 5000x(T − 1)
            end
            }
        function (Calculate expected return of each uniform portfolio) {
            calculate Returns vector (R_p) by summing each column of daily uniform portfolio return matrix
            The size of vector is 5000x1
            end
            }
        function (Estimate the Hurst exponents of uniform portfolios) {
            calculate The Hurst exponents vector (H_p) for each uniform portfolio using Equation (7)
            The size of vector is 5000x1
            end
            }
        Scattered Plot (Returns vector (R_p), Hurst exponents vector (H_p))
end
```

Appendix B

Algorithm: Generating Frontier Portfolios
P_t : The daily price of individual stock at the time $t, t = 1, \ldots, T$
N : Total number of individual stocks
w_i : The portfolio weight of i^{th} stock

```
Begin
        function (Generate returns of individual stocks (R_t)) {
            calculate Returns matrix (R_t) using Equation (11),
            The size of matrix is Nx(T − 1)
            end
            }
        function (Create weight matrix from MV optimization) {
            for sum(w_i) = 1
            calculate 5000 frontier weighs for each individual stock within the range of [0,1]
            end
            }
            The size of matrix is 5000xN
            end
            }
        function (Calculate frontier portfolio return series) {
            calculate Daily frontier portfolio return matrix by R_t × w_i
            The size of matrix is 5000x(T − 1)
            end
            }
        function (Calculate expected return of each frontier portfolio) {
            calculate Returns vector (R_p) by summing each column of daily frontier portfolio return matrix
            The size of vector is 5000x1
            end
            }
```

Appendix B: Continued.

function (Estimate the Hurst exponents of frontier portfolios) {
    calculate The Hurst exponents vector (H_p) for each frontier portfolio using Equation (7)
    The size of vector is $5000x1$
    end
    }
Scattered Plot (Returns vector (R_p), Hurst exponents vector (H_p))
function (Estimate the Lyapunov exponents of frontier portfolios) {
    calculate The Lyapunov exponents vector (λ_p) for each frontier portfolio using
    Equation (10)
    The size of vector is $5000x1$
Scattered 3DPlot (Returns vector (R_p), Hurst exponents vector (H_p),
    Lyapunov exponents vector (λ_p))

end